Naturally graded Lie algebras of slow growth

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$\mathbb{N}$-graded Lie algebras

Definition

A Lie algebra $\mathfrak{g}$ is called $\mathbb{N}$-graded if there is a decomposition

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{N}} \mathfrak{g}_i, \ [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \text{ for all } i, j \in \mathbb{N}.$$

Examples.

We define a $\mathbb{N}$-graded Lie algebra $\mathfrak{g}$ by its infinite basis $e_1, e_2, \dots, e_n, \dots$ and commutation relations.

- a Lie algebra $\mathfrak{m}_0$: $[e_1, e_i] = e_{i+1}, \ \forall \ i \geq 2$.
- the positive part W^+ of the Witt algebra: $[e_i, e_j] = (j-i)e_{i+j}, \ i, j \in \mathbb{N}$.

Narrow graded Lie algebras

Definition

A $\mathbb{N}$-graded Lie algebra $\mathfrak{g} = \bigoplus_{i \in \mathbb{N}} \mathfrak{g}_i$ is called of width d if there exists (minimal) $d \in \mathbb{N}$ such that

$$\dim \mathfrak{g}_i \leq d, \forall i \in \mathbb{N}.$$

The Lie algebra $\mathfrak{m}_0$ considered above and the positive part of the Virasoro algebra Vir^+ are examples of narrowest graded Lie algebras (with width $d = 1$).

Fialowski in 1983 classified $\mathbb{N}$-graded Lie algebras of width 1. Besides $\mathfrak{m}_0$ and W^+ there are two interesting $\mathbb{N}$-graded Lie algebras in her list.

The Lie algebra $\mathfrak{n}_1$

Polynomial matrices defined for $k \in \mathbb{N}$ by

$$e_{3k+1}=\frac{1}{2}\begin{pmatrix} 0 & t^{2k+1} \\ 0 & 0 \end{pmatrix}, e_{3k+2}=\begin{pmatrix} 0 & 0 \\ t^{2k+1} & 0 \end{pmatrix}, e_{3k+3}=\frac{1}{2}\begin{pmatrix} t^{2k+2} & 0 \\ 0 & -t^{2k+2} \end{pmatrix}$$

The linear span $\langle e_1, e_2, e_3, \dots, e_n, \dots \rangle$ is a positively graded subalgebra $\mathfrak{n}_1$ in the loop Lie algebra $\mathfrak{s}l(2, \mathbb{K}) \otimes \mathbb{K}[t]$. It is $\mathbb{N}$-graded with one-dimensional homogeneous components:

$$\mathfrak{n}_1 = \oplus_{i=1}^{+\infty} \langle e_i \rangle \subset \mathfrak{s}l(2, \mathbb{K}) \otimes \mathbb{K}[t],$$

with the Lie bracket

$$[e_i, e_j] = c_{i,j} e_{i+j}, \; c_{i,j} = \begin{cases} 1, \text{if } j - i \equiv 1 \text{ mod} 3; \\ 0, \text{if } j - i \equiv 0 \text{ mod} 3; \\ -1, \text{if } j - i \equiv -1 \text{ mod} 3. \end{cases}$$

Twisted loop algebra $\mathfrak{n}_2 = \oplus_{i=1}^{+\infty}\langle f_i\rangle \subset \mathfrak{sl}(3,\mathbb{K}) \otimes \mathbb{K}[t]$,

$$f_{8k+1}=\begin{pmatrix}0 & t^{2k} & 0\\ 0 & 0 & t^{2k}\\ 0 & 0 & 0\end{pmatrix}, f_{8k+2}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ t^{2k+1} & 0 & 0\end{pmatrix}, f_{8k+3}=\begin{pmatrix}0 & 0 \\ t^{2k+1} & 0 \\ 0 & -t^{2k+1}\end{pmatrix}$$

$$f_{8k+4}=\begin{pmatrix}t^{2k+1} & 0 & 0\\ 0 & -2t^{2k+1} & 0\\ 0 & 0 & t^{2k+1}\end{pmatrix}, f_{8k+5}=\begin{pmatrix}0 & t^{2k+1} & 0\\ 0 & 0 & -t^{2k+1}\\ 0 & 0 & 0\end{pmatrix},$$

$$f_{8k+6}=\begin{pmatrix}0 & 0 & t^{2k+1}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, f_{8k+7}=\begin{pmatrix}0 & 0 & 0\\ t^{2k+2} & 0 & 0\\ 0 & t^{2k+2} & 0\end{pmatrix}, f_{8k+8}=\begin{pmatrix}t^{2k+2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$[f_q, f_l] = d_{q,l} f_{q+l},\ \ q, l \in \mathbb{N}.$$

Structure constants for $\mathfrak{n}_2$.

	f_{8j}	f_{8j+1}	f_{8j+2}	f_{8j+3}	f_{8j+4}	f_{8j+5}	f_{8j+6}	f_{8j+7}
f_{8i}	0	1	-2	-1	0	1	2	-1
f_{8i+1}	-1	0	1	1	-3	-2	0	1
f_{8i+2}	2	-1	0	0	0	1	-1	0
f_{8i+3}	1	-1	0	0	3	-1	1	-2
f_{8i+4}	0	3	0	-3	0	3	0	-3
f_{8i+5}	-1	2	-1	1	-3	0	0	-1
f_{8i+6}	-2	0	1	-1	0	0	0	1
f_{8i+7}	1	-1	0	2	3	1	-1	0

$\mathfrak{n}_1, \mathfrak{n}_2$ as subalgebras of Kac-Moody algebras

- $\mathfrak{n}_1$ is the nilpotent part of the Kac-Moody algebra $A_1^{(1)}$ that corresponds to the generalized Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$: $\mathfrak{n}_1$ is generated by e_1, e_2 subject to two relations
$$ad^{-a_{ij}+1}e_i(e_j) = 0: \; ad^3e_1(e_2) = 0, \; ad^3e_2(e_1) = 0;$$
- $\mathfrak{n}_2$ is the nilpotent part of the Kac-Moody algebra $A_2^{(2)}$ that corresponds to the generalized Cartan matrix $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$: $\mathfrak{n}_1$ is generated by e_1, e_2 subject to two relations
$$ad^{-a_{ij}+1}e_i(e_j) = 0: \; ad^5e_1(e_2) = 0, \; ad^2e_2(e_1) = 0;$$

Descending central series and natural grading

Let $\mathfrak{g}$ be a Lie algebra and its descending central series is

$$\mathfrak{g}^1=\mathfrak{g} \supset \mathfrak{g}^2=[\mathfrak{g},\mathfrak{g}] \supset \dots \supset \mathfrak{g}^k=[\mathfrak{g},\mathfrak{g}^{k-1}] \supset \dots$$

$\mathfrak{g}$ is called nilpotent if there exists s such that $\mathfrak{g}^s \neq 0, \mathfrak{g}^{s+1}=0$.

One can consider its associated graded Lie algebra

$$\mathrm{gr}_C\mathfrak{g} = \oplus_{i=1}^{+\infty} \left(\mathfrak{g}^i/\mathfrak{g}^{i+1}\right)$$

with the Lie bracket:

$$\left[x+\mathfrak{g}^{i+1}, y+\mathfrak{g}^{j+1}\right] = [x,y]+\mathfrak{g}^{i+j+1}, x \in \mathfrak{g}^i, y \in \mathfrak{g}^j.$$

Definition

A Lie algebra $\mathfrak{g}$ is called naturally graded if it is isomorphic to its associated graded $\mathrm{gr}_C\mathfrak{g}$.

Some remarks

The Lie algebra $\mathfrak{m}_0$ is naturally graded:

$$\mathfrak{m}_0 \cong \mathrm{gr}_C \mathfrak{m}_0 = \oplus_{i=1}^{+\infty} \mathfrak{m}_{0i}.$$

But its first homogeneous component is two-dimensional now:

$$\mathfrak{m}_{01} = \langle e_1, e_2 \rangle, \mathfrak{m}_{02} = \langle e_3 \rangle, \dots, \mathfrak{m}_{0i} = \langle e_{i+1} \rangle, i \geq 2.$$

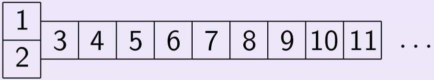
However the positive part W^+ of the Witt algebra is not naturally graded.

$$\mathrm{gr}_C W^+ \cong \mathrm{gr}_C \mathfrak{m}_0 \cong \mathfrak{m}_0.$$

Finite-dimensional naturally graded Lie algebra is called sometimes *Carnot algebra* (Agrachev, Le Donne,...)

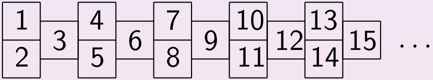
$$\mathfrak{g} = \oplus_{i=1}^{n} \mathfrak{g}_i, \; [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, i = 1, \dots, n-1.$$

$\mathfrak{m}_0$:

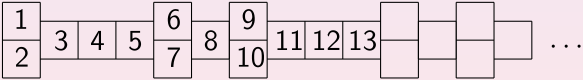


The Lie algebras $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are naturally graded of width $d = 2$.

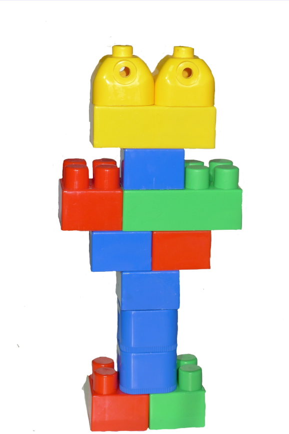
$\mathfrak{n}_1$:



$\mathfrak{n}_2$:



Naturally graded Lie algebras and lego towers



In 1990 O. Mathieu proved V. Kac's conjecture (1968) that an infinite-dimensional $\mathbb{Z}$-graded simple Lie algebra $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ of finite growth, i.e. $\dim \mathfrak{g}_n \leq P(n)$, $P(t) \in \mathbb{R}[t]$, is isomorphic to one of the following types of Lie algebras:

i) (loop algebra) the Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, where $\mathfrak{g}$ is finite-dimensional simple Lie algebra;

ii) (twisted loop algebra) the Lie subalgebra of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$

$$\bigoplus_{\substack{i \in \mathbb{Z},\, i \equiv j \text{ mod n}, \\ j = 0, 1, \dots, n-1}} \mathfrak{g}_j \otimes t^i \subset \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}],$$

where a simple finite-dimensional Lie algebra $\mathfrak{g} = \oplus_{i=0}^{n-1} \mathfrak{g}_i$ is graded by the cyclic group $\mathbb{Z}_n$;

iii) Cartan type Lie algebras: W_n, S_n, K_n, H_n;

iv) the Witt algebra.

Growth of Lie algebras

Suppose that an infinite-dimensional Lie algebra $\mathfrak{g}$ is generated by a finite-dimensional subspace V_1. For $n > 1$, let V^n denote the $\mathbb{K}$-linear span of all products in elements of V_1 of length at most n with arbitrary arrangements of brackets. Clearly $V_1 \subset V_2 \subset \cdots \subset V_n \subset \ldots$ is an ascending chain of finite-dimensional subspaces of $\mathfrak{g}$ and $\cup_{i=1}^{+\infty} V_i = \mathfrak{g}$.
The Gelfand-Kirillov dimension of $\mathfrak{g}$ is

$$GKdim\mathfrak{g} = \limsup_{n \to +\infty} \frac{\log \dim V_n}{\log n}.$$

A finite Gelfand-Kirillov dimension means that there exists a polynomial $P(x)$ such that $\dim V_n < P(n)$ for all $n > 1$.

Growth function $F(n) = \dim V_n$.

- fastest growth – free Lie algebra $L(X)$ with m generators.

$$F(n) \sim \frac{1}{n} m^n$$

- slowest growth – $\mathfrak{m}_0$ and W^+ (maximal class or filiform)

$$F(n) = n+1.$$

- a naturally graded Lie algebra $\mathfrak{g}$ of width d

$$F(n) \leq dn.$$

- $\mathfrak{g} = \mathfrak{n}_1$

$$F(n) = \left[\frac{3n+1}{2}\right].$$

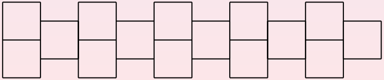
Two real loop algebras

Two real forms $\mathfrak{so}(3,\mathbb{R})$, $\mathfrak{so}(1,2)$ of $\mathfrak{sl}(2,\mathbb{C})$ can be defined by the basis u, v, w and commutating relations

$$[u,v]{=}w,\ [v,w]{=}\pm u,\ [w,u]{=}v.$$

Now we consider two subalgebras $\mathfrak{n}_1^{\pm}$ in loop algebras $\mathfrak{so}(3,\mathbb{R})\otimes\mathbb{R}[t]$ and $\mathfrak{so}(1,2)\otimes\mathbb{R}[t]$ respectively. They are defined by basic elements

$$\begin{matrix} u\otimes t^1 \\ v\otimes t^1 \end{matrix}, w\otimes t^2, \begin{matrix} u\otimes t^3 \\ v\otimes t^3 \end{matrix}, w\otimes t^4, \begin{matrix} u\otimes t^5 \\ v\otimes t^5 \end{matrix}, w\otimes t^6, \dots$$



$\dots \quad \dots$

Theorem (Millionshchikov, Naturally graded Lie algebras (Carnot algebras) of slow growth // arXiv:1705.07494, 2017)

Let $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ *be a real naturally graded Lie algebra such that:*

$$\dim \mathfrak{g}_i + \dim \mathfrak{g}_{i+1} \le 3, \forall i \in \mathbb{N}.$$

Then $\mathfrak{g} = \oplus_{i=1}^{+\infty}\mathfrak{g}_i$ *is isomorphic to the only one Lie algebra from*

$$\mathfrak{m}_0, \mathfrak{n}_1^{\pm},\ \mathfrak{n}_2, \mathfrak{n}_2^3, \left\{\mathfrak{m}_0^S \mid S \subset \{3,5,7,9,\dots\}\right\},$$

where $\mathfrak{n}_2^3$ *is a central extension of* $\mathfrak{n}_2$ *and* $\left\{\mathfrak{m}_0^S \mid S \subset \{3,5,7,9,\dots\}\right\}$ *are central extensions of* $\mathfrak{m}_0$ *that correspond to the sequence* S *of two-cocycles.*

Remark

The Lie algebras $\mathfrak{n}_1^{\pm}$ are isomorphic over $\mathbb{C}$.

Characterisitic Lie algebras (Lie rings) of hyperbolic PDE

- E. Goursat, Annales de la Faculté des Sciences de l'Université de Toulouse 2e serie, **1**:1 (1899), 31–78;
- A.V. Zhiber, A.B. Shabat, Sov.Phys.Dokl. **24**:8 (1979).
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- A.V. Zhiber, R.D. Murtazina, J.Math.Sci., **151**:4 (2008).
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Consider Klein-Gordon non-linear PDE

$$u_{xy} = f(u). \tag{1}$$

Consider a Lie algebra $Diff(\mathcal{F})$ of differential operators acting on the space $\mathcal{F}$ of locally analytic functions on variables $u, u_1, u_2, \dots, u_n, \dots$
Define operator D (the full derivative with respect to x)

$$D = u_1\frac{\partial}{\partial u} + u_2\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2} + \dots + u_{n+1}\frac{\partial}{\partial u_n} + \dots,$$

Define also two operators

$$X_0 = \frac{\partial}{\partial u}, X_1 = f\frac{\partial}{\partial u_1} + D(f)\frac{\partial}{\partial u_2} + \dots + D^{n-1}(f)\frac{\partial}{\partial u_n} + \dots$$

Characteristic Lie ring of a hyperbolic PDE

Consider the commutator

$$[X_0, X_1] = f_u \frac{\partial}{\partial u_1} + D(f_u)\frac{\partial}{\partial u_2} + \dots + D^{n-1}(f_u)\frac{\partial}{\partial u_n} + \dots$$

Are the operators $X_0, X_1, [X_0, X_1]$ functionally independent?
Example:

$$f(u) = e^u, \ [X_0, X_1] = X_1.$$

Hence a linear span $\langle X_0, X_1, [X_0, X_1] \rangle$ for $f(u) = e^u$ is two-dimensional.

Definition

A Lie ring R generated by two operators X_0, X_1 is called characteristic Lie ring of a Klein-Gordon (hyperbolic) equation.

A.V. Zhiber, R.D. Murtazina, On the characteristic Lie algebras for equations "$u_{xy} = f(u, u_x)$.", J. Math. Sci., **151**:4 (2008), 3112–3122.

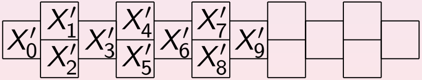
Theorem (M., 2017)

The characteristic Lie algebra of the sinh-Gordon equation

$$u_{xy} = \sinh u$$

is isomorphic to the solvable subalgebra $\tilde{\mathfrak{n}}_1$ of the (non-twisted) affine Lie algebra $A_1^{(1)}$ (Kac-Moody algebra).

$\tilde{\mathfrak{n}}_1$:



... natural grading

A.U. Sakieva, *The characteristic Lie ring of the Zhiber-Shabat-Tzitzeica equation*, Ufa Math.J.**4**:3 (2012),155-160.

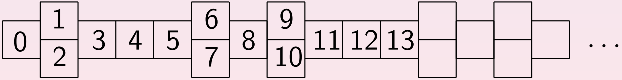
Theorem (M., 2017)

The characteristic Lie algebra of the Tzitzeica equation

$$u_{xy} = e^{u} + e^{-2u}$$

is isomorphic to the solvable Lie subalgebra $\tilde{\mathfrak{n}}_2$ *of the twisted affine Lie algebra* $A_2^{(2)}$.

$\tilde{\mathfrak{n}}_2$:



Theorem (M., 2017)

The characteristic Lie algebra of the sin-Gordon equation

$$u_{xy} = \sin u$$

is isomorphic to the pro-solvable Lie algebra $\tilde{\mathfrak{n}}_1^+$*, the solvable subalgebra of the loop algebra* $\mathfrak{so}(2,1) \otimes \mathbb{R}[t]$*.*

Pro-solvable Lie algebras $\tilde{\mathfrak{n}}_1^{\pm}$ are non-isomorphic over $\mathbb{R}$ and are isomorphic over $\mathbb{C}$ to the solvable subalgebra $\tilde{\mathfrak{n}}_1$ of the Kac-Moody algebra $A_1^{(1)}$.

Thank you!