The space of symmetric squares of hyperelliptic curves: infnite-dimensional Lie algebras and polynomial integrable dynamical systems on $\mathbb{C}^4$

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Motivations

There are quite a few types of infinite dimensional Lie algebras which can be studied in depth, including:

- Witt algebra,
- Kac-Moody algebras,
- Automorphic Lie algebras,
- Polynomial algebras,
- Lie algebras of vector fields tangent to algebraic varieties.

All these infinite dimensional algebras have a faithful representation which can be completely characterised by a finite set of structure elements because they have a structure of a finitely generated module over a certain Noetherian ring (such as a polynomial ring or a coordinate ring of an affine variety).

These algebras have very many applications and are interesting in their own merit.

Outline

- Universal space $\mathrm{Sym}^2(\mathcal{V})$ of the symmetric square of hyperelliptic curves and its coordinate ring.
- Vertical and projectable derivations (vector fields).
- The Newton derivations and Arnold's problem of vector fields on $\mathrm{Sym}^N(\mathbb{C})$.
- Lifting the Witt algebra of the Newton derivations to $\mathrm{Sym}^2(\mathcal{V})$.
- Vertical derivations and extension of the Witt algebra.
- Two commutative vertical derivations and corresponding integrable dynamical systems on $\mathbb{C}^4$.

Symmetric square of hyperelliptic curves

Let integer $N \geq 3$. A hyperelliptic curve $\mathcal{V}_{\mathbf{x}}$ of degree N in $\mathbb{C}^2$ we represent as

$$\mathcal{V}_{\mathbf{x}} = \{(X, Y) \in \mathbb{C}^2 \mid \pi(X, Y) = 0\}$$

where

$$\pi(X, Y) = Y^2 - \prod_{k=1}^{N}(X - x_k)$$

and $\mathbf{x} = (x_1, \ldots, x_N)$ are complex parameters. If $x_i \neq x_j$, $\forall i \neq j$ then the curve is non-singular and of genus $g = \left[\frac{N-1}{2}\right]$.

In $\mathbb{C}^{4+N} = \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^N$ with variables $(X_1, Y_1), (X_2, Y_2), (x_1, \ldots, x_N)$ we consider the affine variety

$$W = \{(X_1, Y_1, X_2, Y_2, x_1, \ldots, x_N) \in \mathbb{C}^{4+N} \mid \pi(X_1, Y_1) = 0,\ \pi(X_2, Y_2) = 0\}.$$

The group $G = S_2 \times S_N$ acts on W by the involution

$$(X_1, Y_1) \longleftrightarrow (X_2, Y_2)$$

and permutations of variables $x_1, \ldots, x_N$.

The universal space $\mathrm{Sym}^2(\mathcal{V}) = W/G$.

The coordinate ring of the symmetric square of hyperelliptic curves

The affine variety W corresponds to the ideal generated by $\pi(X_1, Y_1), \pi(X_2, Y_2)$:

$$J_W = (\pi(X_1, Y_1), \pi(X_2, Y_2)) \subset \mathbb{C}[X_1, Y_1, X_2, Y_2, x_1, \ldots, x_N].$$

The coordinate ring of W is

$$\mathcal{R}_W = \mathbb{C}[X_1, Y_1, X_2, Y_2, x_1, \ldots, x_N]/J_W$$

The coordinate ring of $\mathrm{Sym}^2(\mathcal{V}) = W/G$ is ring of invariants

$$\mathcal{R}_W^G = (\mathbb{C}[X_1, Y_1, X_2, Y_2, x_1, \ldots, x_N]/J_W)^G \simeq (\mathbb{C}[X_1, Y_1, X_2, Y_2, e_1, \ldots, e_N]/J_W^{S_N})^{S_2},$$

where $e_1, \ldots, e_N$ are standard symmetric polynomials

$$e_0 = 1,\ e_1 = x_1 + \cdots + x_N,\ e_2 = \sum_{0<i_1<i_2\leq N} x_{i_1} x_{i_2},\ \ldots, e_N = x_1 x_2 \cdots x_N$$

and the ideal

$$J_W^{S_N} \subset \mathbb{C}[X_1, Y_1, X_2, Y_2, e_1, \ldots, e_N]$$

is generated by $(\pi(X_1, Y_1), \pi(X_2, Y_2))$

$$\pi(X, Y) = Y^2 - \prod_{k=1}^{N}(X - x_k) = Y^2 - \sum_{k=0}^{N}(-1)^k e_k X^{N-k}.$$

The coordinate ring of $\mathrm{Sym}^2(\mathbb{C}^2) = (\mathbb{C}^2 \times \mathbb{C}^2)/S_2$.

Proposition

Let the group S_2 be generated by the involution $(X_1, Y_1) \longleftrightarrow (X_2, Y_2)$ then

$$\mathbb{C}[X_1, Y_1, X_2, Y_2]^{S_2} \simeq \mathbb{C}[u_2, u_4, v_N, v_{N+2}, v_{2N}]/(v_{N+2}^2 - u_4 v_{2N})$$

where

$$u_2 = X_1 + X_2, \quad u_4 = (X_1 - X_2)^2, \quad v_N = Y_1 + Y_2,$$
$$v_{N+2} = (X_1 - X_2)(Y_1 - Y_2), \quad v_{2N} = (Y_1 - Y_2)^2$$

is a basis of the group invariants.

Corollary

A homogeneous polynomial map $\xi : \mathbb{C}^2 \times \mathbb{C}^2 \mapsto \mathbb{C}^5$

$$\xi((X_1, Y_1), (X_2, Y_2)) = (u_2, u_4, v_N, v_{N+2}, v_{2N}),$$

enables us to identify the manifold $\mathrm{Sym}^2(\mathbb{C}^2) = (\mathbb{C}^2 \times \mathbb{C}^2)/S_2$ with a hypersurface in $\mathbb{C}^5$ given by the equation $u_4 v_{2N} - v_{N+2}^2 = 0$.

Grading

All our varieties can be made homogeneous if we assume the following grading weights for the variables

$$|X_i| = 2, \ |Y_i| = N, \ |x_k| = 2, \ |u_k| = |v_k| = k.$$

If we introduce $y_{2k} = e_k$ then

$$|y_s| = s.$$

We can represent $\mathrm{Sym}^2(\mathcal{V})$ as affine variety in the space $\mathbb{C}^{N+5}$ with homogeneous G–invariant coordinates

$$u_2, \ u_4, \ v_N, \ v_{N+2}, \ v_{2N}, \ y_2, \ y_4, \ \ldots, \ y_{2N}.$$

Namely

$$\mathrm{Sym}^2(\mathcal{V}) = V(I) \subset \mathbb{C}^{N+5}.$$

Where

$$I \subset \mathbb{C}[u_2, \ u_4, \ v_N, \ v_{N+2}, \ v_{2N}, \ y_2, \ y_4, \ \ldots, \ y_{2N}]$$

is the ideal defined in the following Proposition

Coordinate ring of the symmetric square of hyperelliptic curves

With $W \subset \mathbb{C}^{N+4}$ we associate its coordinate ring

$$\mathcal{R}_W = \mathbb{C}[X_1, X_2, Y_1, Y_2, x_1, \ldots, x_N] / J_W.$$

The coordinate ring of $\mathrm{Sym}^2(\mathcal{V})$ is the G–invariant subring $\mathcal{R}_W^G \subset \mathcal{R}_W$.

Proposition

The ring $\mathcal{R}_W^G$ is isomorphic to the graded ring

$$\mathcal{R}_I = \mathbb{C}[u_2, u_4, v_N, v_{N+2}, v_{2N}, \mathbf{y}]/I,$$

where $\mathbf{y} = (y_2, y_4, \ldots, y_{2N})$ and the ideal I has Gröbner basis

$$P_{2N+4} = v_{N+2}^2 - u_4 v_{2N},$$

$$P_{2N+2} = v_N v_{N+2} - u_4 \left(a_{2N-2} + \sum_{k=1}^{N-1} (-1)^k y_{2k} a_{2(N-k-1)} \right),$$

$$P_{2N} = v_N^2 + v_{2N} - a_{2N} + u_2 a_{2(N-1)} - \sum_{k=1}^{N-1} (-1)^k y_{2k} (2a_{2(N-k)} - u_2 a_{2(N-k-1)}) - (-1)^N 2 y_{2N},$$

$$P_{3N} = v_N v_{2N} - v_{N+2} \left(a_{2(N-1)} + \sum_{k=1}^{N-1} (-1)^k y_{2k} a_{2(N-k-1)} \right).$$

and the polynomials $a_{2k} = a_{2k}(u_2, u_4)$ of weight $|a_{2k}| = 2k$ are generated by

$$\frac{4}{(2 - u_2 t)^2 - u_4 t^2} = \sum_{k=0}^{\infty} a_{2k}(u_2, u_4) t^k = 1 + u_2 t + \frac{1}{4} \left(3u_2^2 + u_4 \right) t^2 + \frac{1}{2} \left(u_2^3 + u_2 u_4 \right) t^3 + \cdots .$$

⊛ Thus $\mathrm{Sym}^2(\mathcal{V}) = \{(u_2, u_4, v_N, v_{N+2}, v_{2N}, \mathbf{y}) \in \mathbb{C}^{N+5} \,|\, I = 0\}$.
If we allow ourself to divide by $u_4 = (X_1 - X_2)^2$, than we can resolve the system of equations $P_{2N+4} = P_{2N+2} = P_{2N} = P_{3N} = 0$ and explicitly express variables y_{2N-2}, y_{2N} and v_{2N} as elements of the ring $\mathbb{C}[u_2, u_4, v_N, v_{N+2}, y_2, \dots y_{2N-4}][u_4^{-1}]$.

Moreover if we introduce a new variable $v_{N-2} = v_{N+2}u_4^{-1}$, then

$$y_{2N-2}, y_{2N}, v_{N+2}, v_{2N} \in \mathbb{C}[u_2, u_4, v_{N-2}, v_N, y_2, \dots y_{2N-4}].$$

It defines the polynomial map $\phi\colon \mathbb{C}^{N+2} \to \mathbb{C}^{N+5}$ defined by

$$\phi : (u_2, u_4, v_{N-2}, v_N, y_2, \dots y_{2N-4}) = (u_2, u_4, v_N, v_{N+2}, v_{2N}, \mathbf{y})$$

$$v_{N+2} = u_4 v_{N-2}, \qquad v_{2N} = u_4 v_{N-2}^2,$$

$$y_{2(N-1)} = (-1)^{N-1}\left(v_N v_{N-2} - a_{2(N-1)} - \sum_{k=2}^{N-2}(-1)^k y_{2k} a_{2(N-k-1)}\right),$$

$$y_{2N} = \frac{(-1)^N}{2}\left[v_N^2 + v_{2N} - 2a_{2N} + u_2 a_{2(k-1)} - \sum_{k=2}^{N-1}(-1)^k y_{2k}(2a_{2(N-k)} - u_2 a_{2(N-k-1)})\right].$$

Thus, $\mathrm{Sym}^2(\mathcal{V})$ is bi-rationally isomorphic to $\mathbb{C}^{N+2}$.

Theorem

The mapping ϕ is a bi-rational isomorphism

$$\phi\colon \mathbb{C}^{N+2} \setminus \{u_4 = 0\} \to \mathrm{Sym}^2(\mathcal{V}) \setminus (\{u_4 = 0\} \cap \mathrm{Sym}^2(\mathcal{V})).$$

Short summary:

$$\begin{array}{ccc}
\mathbb{C}[x_1,\ldots,x_N\,;\;X_1,Y_1,X_2,Y_2] & & \mathbb{C}[x_1,\ldots,x_N\,;\;X_1,Y_1,X_2,Y_2]/J_W \\
\bigcup & & \bigcup \\
(\mathbb{C}[x_1,\ldots,x_N\,;\;X_1,Y_1,X_2,Y_2])^{S_N\times S_2} & & (\mathbb{C}[x_1,\ldots,x_N\,;\;X_1,Y_1,X_2,Y_2]/J_W)^{S_N\times S_2} \\
\wr| & & \wr| \\
\mathbb{C}[y_2,\ldots,y_{2N}\,;\;u_2,u_4,v_N,v_{N+2},v_{2N}]/(\mathrm{syz}) & & \mathbb{C}[y_2,\ldots,y_{2N}\,;\;u_2,u_4,v_N,v_{N+2},v_{2N}]/I \\
\downarrow & & \downarrow \\
\mathbb{C}[y_2,\ldots,y_{2N},u_2,u_4,v_{N-2},v_N] & & \mathbb{C}[y_2,\ldots,y_{2N-4},u_2,u_4,v_{N-2},v_N]
\end{array}$$

where $\mathrm{syz} = v_{N+2}^2 - u_4 v_{2N} = P_{2N+4}$, $J_W = (\pi(X_1, Y_1), \pi(X_2, Y_2))$ and

$$I = (P_{2N+4}, P_{2N+2}, P_{2N}, P_{3N})$$

$$\mathrm{Sym}^2(\mathcal{V}) = \{(y_2,\ldots,y_{2N}; u_2,u_4,v_N,v_{N+2},v_{2N}) \in \mathbb{C}^{N+5} \mid P_{2N+4} = P_{2N+2} = P_{2N} = P_{3N} = 0\}.$$

Vertical and projectable derivations

Definition

- *A derivation L of a quotient ring* $\mathcal{R} = \mathbb{C}[a_1, \ldots, a_n\,;\; b_1, \cdots, b_m]/J$ *over the ideal J is a derivation of the ring* $\mathcal{R} = \mathbb{C}[a_1, \ldots, a_n\,;\; b_1, \cdots, b_m]$ *such that* $L(J) \subseteq J$.
- *A derivation L is called* **vertical***, if* $L(a_i) \in J,\ i = 1, \ldots, n$.
- *There is a canonical homomorphism* $j_* : \mathbb{C}[a_1, \ldots, a_n] \to \mathcal{R}$. *A derivation L of* $\mathcal{R}$ *is called* **projectable** *with the projection* $\hat{L}$*, if there exists a derivation* $\hat{L}$ *of the ring* $\mathbb{C}[a_1, \ldots, a_n]$ *such that*

$$L(j_*(a_i)) = j_*(\hat{L}(a_i)), \qquad i = 1, \ldots, n.$$

Thus vertical derivations are represented by the vector fields of the form

$$L = \sum_{i=1}^{m} B_i \frac{\partial}{\partial b_i}, \qquad B_i \in \mathcal{R}.$$

projectable derivations are of the form

$$L = \sum_{i=1}^{n} A_i \frac{\partial}{\partial a_i} + \sum_{i=1}^{m} B_i \frac{\partial}{\partial b_i}, \qquad A_i \in \mathbb{C}[a_1, \ldots, a_n] \cap \mathcal{R}, \;\; B_i \in \mathcal{R}.$$

In both cases we assume that $L(J) \subset J$.

Arnold's problem of vector fields tangent to a discriminant.

The problem of construction of vector fields in $\mathbb{C}^N,\ x_1+\cdots+x_N=0$ which are tangent to the discriminant set

$$\mathcal{D}=\{(x_1,\ldots,x_N)\in\mathbb{C}^N\,|\,\Delta=0\},\qquad \Delta=\prod_{i<j}(x_i-x_j)^2$$

have been solved by Arnold and his group in 1976-1980 (see V.Arnold Singularities of Caustics and Wave Fronts 1996). D.Fuks proposed the method to compute the vector fields using the convolution algebra. Then V.Zakalyukin has shown that there exists a basis of vector fields such that the derivations $L_0^A\ldots,L_{N-2}^A$ acting on the standard symmetric polynomials results in a symmetric matrix

$$L_{k-2}^A(e_m)=L_{m-2}^A(e_k).$$

Here we will give a new and short method to solve this problem, as well as we prove Eilbeck's conjecture that

$$L_k^A(\Delta)=(N-k)(N-k-1)e_k\Delta$$

The Newton derivations of $\mathcal{R}_N = \mathbb{C}[x_1, \ldots, x_N]^{S_N}$

Problem (Arnold): Find derivations L_k^A of the polynomial ring $\mathbb{C}[x_1, \ldots, x_N]$ such that

$$L_k^A : \mathbb{C}[x_1, \ldots, x_N]^{S_N} \mapsto \mathbb{C}[x_1, \ldots, x_N]^{S_N}$$
$$L_k^A : (\Delta) \mapsto (\Delta),$$
$$L_{k-2}^A(e_m) = L_{m-2}^A(e_k),$$
$$L_k^A : (x_1 + \cdots + x_N) \mapsto (x_1 + \cdots + x_N).$$

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Newton polynomials:

$$p_k = \sum_{i=1}^{N} x_i^k, \qquad k = 0, 1, 2, \ldots$$

$$p_0 = N,\ p_1 = e_1 = x_1 + \cdots + x_N,\ p_2 = x_1^2 + \cdots + x_N^2, \ldots$$

The set $p_1, \ldots, p_N$ form a basis in the ring of S_N invariants:

$$\mathcal{R}_N = \mathbb{C}[x_1, \ldots, x_N]^{S_N} \simeq \mathbb{C}[e_1, \ldots, e_N] \simeq \mathbb{C}[p_1, \ldots, p_N],$$
$$p_{N+k} = p_{N+k}(p_1, \ldots, p_N), \qquad k = 1, 2, \ldots$$

Generating function for Newton's polynomials

$$\mathcal{N}(t) = \sum_{k=0}^{\infty} p_k t^k = \sum_{i=1}^{N} \frac{1}{1 - t x_i}.$$

The Newton derivations of $\mathcal{R}_N = \mathbb{C}[x_1, \ldots, x_N]^{S_N}$

Definition

The derivations of the ring $\mathbb{C}[x_1, \ldots, x_N]$ *of the form*

$$L_q^0 = \sum_{i=1}^{N} x_i^{q+1} \partial_{x_i}, \qquad q = -1, 0, 1, \ldots,$$

are called the Newton derivations

Proposition

Newton derivations of $\mathbb{C}[x_1, \ldots, x_N]$

- *map symmetric polynomials into symmetric*

$$L_k^0 : \mathbb{C}[x_1, \ldots, x_N]^{S_N} \mapsto \mathbb{C}[x_1, \ldots, x_N]^{S_N}, \quad L_k^0(p_n) = np_{k+n},$$

- *give a faithful representation of the Witt algebra*

$$[L_m^0, L_n^0] = (n - m)L_{n+m}^0,$$

- *map the discriminant ideal into itself*

$$L_n^0(\Delta) = \gamma_n^0 \Delta, \qquad \gamma_n^0 \in \mathbb{C}[x_1, \ldots, x_N]^{S_N}.$$

Corollary

For all $k, q \in \mathbb{N}$ and $n = 1, \ldots,$ the polynomials p_k, $k = 0, 1, \ldots,$ are related by

$$\sum_{m=1}^{N} m\left(p_{(k+m)}\frac{\partial p_{(q+n)}}{\partial p_m} - p_{(q+m)}\frac{\partial p_{(k+n)}}{\partial p_m}\right) = (q-k)p_{(k+q+n)}.$$

Only the first N derivations L^0_k, $k = -1, 0, 1, \ldots, N-2$ are linearly independent over $\mathcal{R}_N$:

$$L^0_n = \sum_{s=1}^{N} w_{n,s} L^0_{s-2}, \qquad w_{n,s} \in \mathcal{R}_N.$$

They are generators of a free left $\mathcal{R}_N$–module.

Generating derivations

It is convenient to introduce the generating derivation

$$L^0(t) = \sum_{k=0}^{\infty} t^k L^0_{k-1} = \sum_{i=1}^{N} \frac{1}{1 - x_i t} \frac{\partial}{\partial x_i}$$

Then it is easy to verify that

$$L^0(t)(E(\tau)) = -\tau E(\tau) \sum_{i=1}^{N} \frac{1}{(1 - x_i t)(1 - x_i \tau)}$$

where $E(\tau) = \sum_{k=0}^{N} (-\tau)^k e_k = \prod_{i=1}^{N} (1 - x_i \tau)$. Thus

$$tE(t)L^0(t)(E(\tau)) = -t\tau E(t)E(\tau) \sum_{i=1}^{N} \frac{1}{(1 - x_i t)(1 - x_i \tau)}$$

and therefore the derivations $\hat{L}^A_k$ generating by the derivation

$$\hat{L}^A(t) = tE(t)L^0(t) = \sum_{k=1}^{N} (-1)^k \hat{L}^A_{k-2} t^k$$

yield a symmetric matrix $\hat{L}^A_{k-2}(e_m) = \hat{L}^A_{m-2}(e_k)$.
Derivations $\hat{L}^A_k,\ k = -1, 0, 1, \ldots, N-2$ have properties:

$$\hat{L}^A_{k-2} : \mathcal{R}_N \mapsto \mathcal{R}_N; \quad \hat{L}^A_{k-2} : (\Delta) \mapsto (\Delta); \quad \hat{L}^A_{k-2}(e_m) = \hat{L}^A_{m-2}(e_k).$$

Generating derivation for Arnold's vector fields and Eibeck's conjecture

The missing property (if we wish fit exactly Arnold's derivation) is that

$$\hat{L}^A_{k-2}(x_1 + \cdots + x_N) \not\subset (x_1 + \cdots + x_N).$$

It can be easily corrected: the generating derivation

$$L^A(t) = \hat{L}^A(t) + N^{-1} L^0_{-1}(E(t)) L^0_{-1} = \sum_{m=2}^{N} (-1)^m t^m L^A_{m-2}$$

generates Arnold's derivations.

Conjecture (Eilbeck)

The discriminant polynomial $\Delta = \prod_{i<j}(x_i - x_j)^2$ *is an eigenvector of Arnold's derivations*

$$L^A_k \Delta = \gamma^A_k \Delta, \qquad k = 0, 1, \ldots, N-2$$

where

$$\gamma^A_k = (N-k)(N-k-1)e_k, \quad k = 0, 1, 2, \ldots N-2.$$

Proof:

$$\sum_{m=2}^{N} (-1)^m t^m L^A_{m-2} \Delta = L^A(t)\Delta = \gamma^A(t)\Delta,$$

where

$$\gamma^A(t) = t^2(t^2 E_{tt}(t) - 2t(N-1)E_t(t) + N(N-1)E(t)) = t^2 \sum_{k=0}^{N-2} (-1)^k t^k (N-k)(N-k-1) e_k.$$

Projectable tangent vector fields on $\mathrm{Sym}^2(\mathcal{V})$

Newton derivations

$$\mathcal{L}^0(t) = \sum_{k=0}^{\infty} t^k \mathcal{L}^0_{2k-2} = \sum_{i=1}^{N} \frac{2}{1 - tx_i} \frac{\partial}{\partial x_i}, \quad \mathcal{L}^0_{2q} = 2 \sum_{i=1}^{N} x_i^{q+1} \frac{\partial}{\partial x_i}, \qquad q = -1, 0, 1, \dots,$$

do not represent tangent vector fields to $\mathrm{Sym}^2(\mathcal{V})$

$$\mathcal{L}^0_{2q}(J_W) \not\subset J_W.$$

They can be "corrected":

Proposition

There is a unique lift of the Newton derivations $\mathcal{L}^0_{2k}$ sach that the vector fields $\mathcal{L}_{2k}, \; k = -1, 0, 1, \dots$ are:

tangent to $\mathrm{Sym}^m(\mathcal{V})$:	$\mathcal{L}_{2k}(J_W) \subset J_W,$
represent the Witt algebra:	$[\mathcal{L}_{2k}, \mathcal{L}_{2n}] = 2(n-k)\mathcal{L}_{2(n-k)}.$

The derivations $\mathcal{L}_{2k}$ are generated by

$$\mathcal{L}(t) = \sum_{k=0}^{\infty} t^k \mathcal{L}_{2k-2} = \sum_{i=1}^{N} \frac{2}{1 - tx_i} \frac{\partial}{\partial x_i} + \sum_{s=1}^{m} \left(\frac{2}{1 - tX_s} \frac{\partial}{\partial X_s} + \frac{tY_s \mathcal{N}(t)}{1 - tX_s} \frac{\partial}{\partial Y_s} \right)$$

where $\mathcal{N}(t)$ is the generating function of the Newton polynomials

$$\mathcal{N}(t) = \sum_{i=1}^{N} \frac{1}{1 - tx_i}.$$

Commuting vector fields of symmetric square of a curve.

Lemma

Let $F(X, Y)$ be a twice differentiable function. Let D_k be defined as

$$D_k = \partial_{Y_k}(F(X_k, Y_k))\partial_{X_k} - \partial_{X_k}(F(X_k, Y_k))\partial_{Y_k}, \qquad k = 1, 2.$$

and

$$\mathcal{L}^{*1} = \frac{D_1 - D_2}{X_1 - X_2}, \qquad \mathcal{L}^{*2} = \frac{X_2 D_1 - X_1 D_2}{X_1 - X_2}.$$

Then the vector fields $\mathcal{L}^{*1}, \mathcal{L}^{*2}$

- *commute $[\mathcal{L}^{*1}, \mathcal{L}^{*2}] = 0$,*
- *map symmetric $(X_1, Y_1) \leftrightarrow (X_2, Y_2)$ functions into symmetric,*

and functions $F(X_k, Y_k),\ k = 1, 2$ are in their kernel space $(\mathcal{L}^i(F(X_j, Y_j)) = 0)$.

Lemma

Let all roots of the polynomial $P(X)$ be distinct and $J_\pi = (Y^2 - P(X))$ be the ideal in $\mathbb{C}[X, Y]$. Then any derivation D of the quotient ring $\mathbb{C}[X, Y]/J_\pi$ can be represented in the form $D = aD^\star$, where $a \in \mathbb{C}[X, Y]/J_\pi$ and

$$D^\star = 2Y\partial_X + \partial_X(P(x))\partial_Y.$$

Commuting vector fields on $\mathrm{Sym}^2(\mathcal{V})$.

Taking $F(X_j, Y_j) = \pi_j$ we obtain two commuting vector fields on $\mathrm{Sym}^2(\mathcal{V})$ (and derivations on the corresponding coordinate rings).

What is **surprising** is that $\mathcal{L}^*_{N-4} = \mathcal{L}^{*1}, \mathcal{L}^*_{N-2} = \mathcal{L}^{*2}$ are polynomial derivations in variables u_2, u_4, v_{N-2}, v_N.

Example: For example in the case $N = 5$ we get

$$\mathcal{L}^*_1 = 4u_3\partial_{u_2} + (5u_2^2 + u_4 + 2y_4)\partial_{u_3} + 8u_5\partial_{u_4} + (5u_2^3 + 5u_2u_4 + 6u_2y_4 - 4y_6)\partial_{u_5}$$

$$\mathcal{L}^*_3 = 4(u_5 - u_2u_3)\partial_{u_2} - 4(u_3^2 - u_2u_4 - u_2y_4 + y_6)\partial_{u_3} + 8(u_3u_4 - u_2u_5)\partial_{u_4} + (u_4^2 - 5u_2^4 + 4u_3u_5 - 6u_2^2y_4 + 2u_4y_4 + 4u_2y_6)\partial_{u_5}$$

Proposition

Derivations $\mathcal{L}_0, \mathcal{L}_2, \mathcal{L}_4, \mathcal{L}_6$ and $\mathcal{L}^*_1, \mathcal{L}^*_3$ form a polynomial Lie algebra, which is isomorphic to the polynomial Lie algebra of vector fields on the Jacobian of genus 2.

Now it is not surprising that u_2 provides algebra-geometric solution to the KdV equation. Denoting $u := 8u_2,\ (\mathcal{L}^*_1)^k(u) = \partial_x^k u, \mathcal{L}^*_3(u) = \partial_t u$ we get

$$8u_t = u_{xxx} - 6uu_x,$$
$$128y_4u_x + u_{xxxxx} - 10uu_{xxx} - 20u_xu_{xx} + 30u^2u_x = 0.$$

Polynomial integrable dynamical systems in $\mathbb{C}^4$

Commuting vector fields $\Leftrightarrow$ compatible dynamical systems.
$N = 3$ – Elliptic case:

$$Y^2 = X^3 + \lambda_4 X + \lambda_6.$$

Let $\mathcal{L}^*_{-1} f = f'$ and $\mathcal{L}^*_1 f = \dot{f}$.
We get two dynamical systems:

$$u_2' = 2v_1; \qquad u_4' = 4v_3; \qquad v_1' = 1; \qquad v_3' = 3u_2;$$

$$\dot{u}_2 = u_2 v_1 - v_3; \qquad \dot{u}_4 = -2(u_4 v_1 - u_2 v_3); \qquad \dot{u}_1 = -u_2 + v_1^2;$$

$$\dot{u}_3 = \frac{1}{2}(3u_2^2 - u_4 - 2v_1 v_3).$$

They commute and have two common first integrals:

$$\lambda_4 = v_1 v_3 - \frac{1}{4}(3u_2^2 + u_4);$$

$$\lambda_6 = \frac{1}{4}(v_3^2 - 2u_2 v_1 v_3 + u_2^3 - u_2 u_4 + u_4 v_1^2).$$

Polynomial integrable dynamical systems in $\mathbb{C}^4$

In the original coordinates (X_1, Y_1, X_2, Y_2) we get rational dynamical systems

$$X_1' = 2\frac{Y_1}{X_1 - X_2}; \qquad Y_1' = \frac{3X_1^2 + \lambda_4}{X_1 - X_2};$$

$$X_2' = 2\frac{Y_2}{X_2 - X_1}; \qquad Y_2' = \frac{3X_2^2 + \lambda_4}{X_2 - X_1};$$

$$\dot{X}_1 = 2\frac{X_2 Y_1}{X_1 - X_2}; \qquad \dot{Y}_1 = \frac{X_2(3X_1^2 + \lambda_4)}{X_1 - X_2};$$

$$\dot{X}_2 = 2\frac{X_1 Y_2}{X_2 - X_1}; \qquad \dot{Y}_2 = \frac{X_1(3X_2^2 + \lambda_4)}{X_2 - X_1}.$$

which can be integrated in the elliptic functions:

$$X_1 = \wp(z_1 + z_1^0);\ \ Y_1 = \wp'(z_1 + z_1^0);\ \ X_2 = \wp(z_2 + z_2^0);\ \ Y_2 = \wp'(z_2 + z_2^0),$$

where z_1^0 and z_2^0 are arbitrary constants.

The cases $N = 5, 6$ we have $\mathrm{Sym}^2(\mathcal{V}) \simeq \mathrm{Jac}(\mathcal{V})$ and the corresponding dynamical systems can be integrated in the Abelian functions of genus 2. The case $N = 5$ corresponds to Dubrovin's system on the Jacobian for $g = 2$.

Polynomial integrable dynamical system in $\mathbb{C}^4$ for $N = 7$.

The first nontrivial case is $N = 7$ ($g = 3$). The polynomial dynamical system in $\mathbb{C}^4$ corresponding to the curve

$$V_\lambda = \{(X, Y) \in \mathbb{C}^2 \colon Y^2 = X^7 + y_4 X^5 + \ldots - y_{14}\},$$

are of the form:

$$\begin{aligned}
&\mathcal{L}_3^* u_2 = v_5, \qquad \mathcal{L}_3^* u_4 = v_7, \\
&\mathcal{L}_3^* v_5 = 3u_4^2 + 35u_2^4 + 42u_2^2 u_4 + 2y_4(u_4 + 5u_2^2) - 4y_6 u_2 + y_8, \\
&\mathcal{L}_3^* v_7 = 14(3u_2^5 + 3u_2 u_4^2 + 10u_2^3 u_4) + 20y_4(u_2 u_4 + u_2^3) - \\
&\qquad - 4y_6(u_4 + 3u_2^2) + 6y_8 u_2 - 2y_{10}.
\end{aligned}$$

$$\begin{aligned}
&\mathcal{L}_5^* u_2 = u_2 v_5 - \frac{1}{2} v_7, \qquad \mathcal{L}_5^* u_4 = u_2 v_7 - 2u_4 v_5, \\
&\mathcal{L}_5^* v_5 = v_5^2 + 14u_2^5 - 18u_2 u_4^2 - 28u_2^3 u_4 - 8y_4 u_2 u_4 + 2y_6(u_2^2 + u_4) - \\
&\qquad - 2y_8 u_2 + y_{10}, \\
&\mathcal{L}_5^* v_7 = 35u_2^6 - 63u_2^2 u_4^2 + 35u_2^4 u_4 - 7u_4^3 + 5y_4(3u_2^4 - 2u_2^2 u_4 - u_4^2) - \\
&\qquad - 8y_6(u_2^3 - u_2 u_4) + 3y_8(u_2^2 - u_4) - y_{12}.
\end{aligned}$$

These two compatible systems possess two common first integrals:

$$y_{12} = v_5 v_7 - (7u_2^6 + 35u_2^4 u_4 + u_4^3 + 21u_2^2 u_4^2) -$$
$$- y_4(5u_2^4 + u_4^2 + 10u_2^2 u_4) + 4y_6(u_2^3 + u_2 u_4) -$$
$$- y_8(3u_2^2 + u_4) + 2y_{10} u_2,$$

$$y_{14} = -\frac{1}{4} v_7^2 - u_4 v_5^2 + u_2^7 + 21u_2^5 u_4 + 35u_2^3 u_4^2 + 7u_2 u_4^3 +$$
$$+ y_4(u_2^5 + 10u_2^3 u_4 + 5u_2 u_4^2) - y_6(u_2^4 + u_4^2 + 6u_2^2 u_4) +$$
$$+ y_8(u_2^3 + 3u_2 u_4) - y_{10}(u_2^2 + u_4) + y_{12} u_2.$$

These systems can be integrated in meromorphic functions on σ–divisor of the hyperelliptic curve of genus $g = 3$ [recent paper by Victor Buchstaber and Takanori Ayano]. They cannot be integrated in the Abelian $2g = 6$ periodical functions! A new class of functions is required.

Summary

Lie algebras of derivations of Abelian functions originally had been constructed using the theory of multi–dimensional σ functions (Buchstaber, Leykin). It was based on the analytic theory of hyperelliptic Jacobians and θ functions (Dubrovin, Novikov). They did not use the algebraic geometry of the symmetric powers of hyperelliptic curves.

We have shown that using the algebraic geometry of the symmetric powers of hyperelliptic curves one can obtain differential equations integrable in terms of Abelian functions and functions meromorphic on the σ–divisor without using the properties of these functions for the construction of the equations.

We have drastically simplified and developed further Arnold's theory of vector fields tangent to the singularities (the discriminant) in his approach to the problem of wave fronts and caustics. We have proved Eilbeck's Conjecture concerning the eigenvalues of Arnold's vector fields on the discriminant.

We have constructed graded Lie algebras of vector fields, such that projectable vector fields are tangent to the discriminant of the curve which is a sub–manifold in the space of the curve parameters, while the vertical vector fields are tangent to the curves with fixed parameters. We have effectively solved a nontrivial problem of lifting of projectable vector fields from the parameters space to the whole variety of the symmetric power of the curves (explicit and pure algebraic construction of the Gauss-Manin connection).