Integrable vector equations

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Historical remarks.

Symmetry approach to classification of integrable PDEs was developed by: A.Shabat, A.Zhiber, N.Ibragimov, A.Fokas, V.Sokolov, S.Svinolupov, A.Mikhailov, R.Yamilov, V.Adler, P.Olver, J.Sanders, J.P.Wang, V.Novikov, A.Meshkov, D.Demskoy, H.Chen, Y.Lee, C.Liu, I.Khabibullin, B.Magadeev, R.Heredero, V.Marikhin, M.Foursov, S.Startcev, M.Balakhnev, ...

Definition. PDE is integrable if it possesses infinitely many generalized infinitesimal symmetries.

The first classification result in frames of the symmetry approach was obtained by **A. Zhiber and A. Shabat, 1979**:

Theorem. Nonlinear hyperbolic equation of the form

$$u_{xy} = F(u)$$

possesses higher symmetries iff (up to scalings and shifts)

$$F(u) = e^u, \quad F(u) = e^u + e^{-u}, \quad \text{or} \quad F(u) = e^u + e^{-2u}.$$

Integrable equations of KdV-type were described in

Theorem (S. Svinolupov-VS 1982). A complete list (up to "‘almost invertible"’ transformations) of non-linear equations of the form

$$u_t = u_{xxx} + f(u, u_x, u_{xx}) \qquad (1)$$

possessing infinite hierarchy of symmetries and conservation laws can be written as:

$$u_t = u_{xxx} + 6u\, u_x, \qquad \text{(KdV)}$$

$$u_t = u_{xxx} + 6u^2\, u_x, \qquad \text{(mKdV)}$$

$$u_t = u_{xxx} - \frac{1}{2}u_x^3 + (\alpha e^{2u} + \beta e^{-2u})u_x, \qquad \text{(CD1)}$$

$$u_t = u_{xxx} - \frac{1}{2}Q''\, u_x + \frac{3}{8}\,\frac{(Q - u_x^2)_x^2}{u_x\,(Q - u_x^2)}, \qquad \text{(CD2)}$$

$$u_t = u_{xxx} - \frac{3}{2}\,\frac{u_{xx}^2 + Q(u)}{u_x}, \qquad \text{(KN)}$$

where $Q^{(5)}(u) = 0$.

All equations of the form

$$u_t = u_{xxxxx} + F(u, u_x, u_{xx}, u_{xxx}, u_{xxxx}),$$

possessing higher conservation laws were found by **V. Drinfeld, VS, S. Svinolupov, 1985**.

Classification of two-component systems

The most significant work has been done by **A. Mikhailov, A. Shabat, R. Yamilov, 1987**. All systems of the form

$$u_t = u_{xx} + F(u, v, u_x, v_x), \qquad v_t = -v_{xx} + G(u, v, u_x, v_x)$$

possessing higher symmetries and conservation laws were listed.

Example 1. Well-known NLS-equation

$$u_t = u_{xx} + u^2 v, \qquad v_t = -v_{xx} - v^2 u.$$

Example 2. The Landau-Lifshitz equation (after stereographic projection)

$$u_t = u_{xx} - \frac{2u_x^2}{u+v} - \frac{4(\,p(u,v)\,u_x + r(u)\,v_x\,)}{(u+v)^2}$$

$$v_t = -v_{xx} + \frac{2v_x^2}{u+v} - \frac{4(\,p(u,v)\,v_x + r(-v)\,u_x\,)}{(u+v)^2},$$

where

$$r(y) = c_4 y^4 + c_3 y^3 + c_2 y^2 + c_1 y + c_0$$

and

$$p(u,v) = 2c_4 u^2 v^2 + c_3(uv^2 - vu^2) - 2c_2 uv + c_1(u - v) + 2c_0.$$

Integrable matrix evolution equations

P. Olver and VS 1998 listed integrable matix polynomial evolution equations having higher symmetries. One of the lists:

$$u_t = u_{xxx} + 3u^2 u_x + 3u_x u^2,$$
$$u_t = u_{xxx} + 3uu_{xx} - 3u_{xx}u - 6uu_x u,$$
$$u_t = u_{xxx} + 3u_x^2.$$

Here u is an $m \times m$-matrix. Equations are integrable for arbitrary m.

Second order matrix systems of NLS- and DNLS-types also were listed and several new integrable models were found.

Examples:

$$u_t = u_{xx} + 2(u+v)u_x, \qquad v_t = -v_{xx} + 2v_x(u+v);$$

$$u_t = u_{xx} + 2u_x vu, \qquad v_t = -v_{xx} + 2vuv_x.$$

Vector integrable equations.

Example 1. The following vector mKdV-systems:

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + (\boldsymbol{u}, \boldsymbol{u})\, \boldsymbol{u}_x,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + (\boldsymbol{u}, \boldsymbol{u})\, \boldsymbol{u}_x + (\boldsymbol{u}, \boldsymbol{u}_x)\, \boldsymbol{u},$$

where $\boldsymbol{u}$ is N-component vector, are integrable for any N.

Example 2. Integrable vector NLS-system (Manakov):

$$\boldsymbol{u}_t = \boldsymbol{u}_{xx} + (\boldsymbol{u}, \boldsymbol{v})\, \boldsymbol{u}, \qquad \boldsymbol{v}_t = -\boldsymbol{v}_{xx} - (\boldsymbol{u}, \boldsymbol{v})\, \boldsymbol{v}.$$

Example 3. Integrable vector KdV equation (Svinolupov-VS)

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + (\boldsymbol{c}, \boldsymbol{u})\, \boldsymbol{u}_x + (\boldsymbol{c}, \boldsymbol{u}_x)\, \boldsymbol{u} - (\boldsymbol{u}, \boldsymbol{u}_x)\, \boldsymbol{c},$$

where $\boldsymbol{c}$ is an arbitrary constant vector.

The simplest conserved densities of the 3-rd equation are:

$$\rho_1 = (\boldsymbol{c}, \boldsymbol{u}),$$

$$\rho_2 = \boldsymbol{c}^2 \boldsymbol{u}^2 - 2(\boldsymbol{c}, \boldsymbol{u})^2,$$

and

$$\rho_3 = 6(\boldsymbol{c}, \boldsymbol{u}_x)^2 - 3\boldsymbol{c}^2 \boldsymbol{u}_x^2 + 3\boldsymbol{c}^2 \boldsymbol{u}^2 (\boldsymbol{c}, \boldsymbol{u}) - 4(\boldsymbol{c}, \boldsymbol{u})^3.$$

Example 4. Consider equation (**I.Golubchik, VS, 2000**):

$$\boldsymbol{u}_t = \Big(\boldsymbol{u}_{xx} + \frac{3}{2}(\boldsymbol{u}_x,\ \boldsymbol{u}_x)\boldsymbol{u}\Big)_x + \frac{3}{2}(\boldsymbol{u}, R(\boldsymbol{u}))\,\boldsymbol{u}_x, \qquad \boldsymbol{u}^2 = 1, \quad (2)$$

where $R = \mathrm{diag}(r_1, ..., r_N)$ is arbitrary constant matrix.

If $N = 3$, then (2) is a commuting flow of the Landau-Lifshitz equation.

Consider integrable vector evolution equations of the following form:

$$\boldsymbol{u}_t = f_n\,\boldsymbol{u}_n + f_{n-1}\,\boldsymbol{u}_{n-1} + \cdots + f_1\,\boldsymbol{u}_1 + f_0\,\boldsymbol{u}, \quad \boldsymbol{u}_i = \frac{\partial^i \boldsymbol{u}}{\partial x^i}. \quad (3)$$

Here $\boldsymbol{u}$ is N-component vector, the (scalar) coefficients f_i depend on scalar products between $\boldsymbol{u}, \boldsymbol{u}_x, ..., \boldsymbol{u}_n$.

We consider equations (3) that are integrable for arbitrary dimension N. In virtue of the arbitrariness of N, all scalar products

$$u_{[i,j]} = (\boldsymbol{u}_i, \boldsymbol{u}_j), \qquad i \leqslant j$$

can be regarded as functionally independent variables.

We denote the ring of scalar-valued functions depending on finite number of scalar products by $\mathcal{F}$.

The vector mKdV-systems:

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + (\boldsymbol{u}, \boldsymbol{u})\, \boldsymbol{u}_x,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + (\boldsymbol{u}, \boldsymbol{u})\, \boldsymbol{u}_x + (\boldsymbol{u}, \boldsymbol{u}_x)\, \boldsymbol{u}$$

as well as the following integrable vector Harry Dym equation

$$\boldsymbol{u}_t = (\boldsymbol{u}, \boldsymbol{u})^{3/2} \boldsymbol{u}_{xxx}$$

have the form (3). Notice that the coefficients of the latter equation are not polynomial.

It is clear all such equations (3) are isotropic i.e. are invariant with respect to the group $SO(N)$.

Theorem. **A.Meshkov, VS 2002**

i). If equation (3) possesses an infinite series of vector commuting flows of the form

$$\boldsymbol{u}_\tau = g_m\,\boldsymbol{u}_m + g_{m-1}\,\boldsymbol{u}_{m-1} + \cdots + g_1\,\boldsymbol{u}_1 + g_0\,\boldsymbol{u}, \qquad g_i \in \mathcal{F}, \quad (4)$$

then there exists a formal Lax pair $L_t = [A,\, L]$, where

$$L = a_1\,D_x + a_0 + a_{-1}\,D_x^{-1} + \cdots\,, \qquad A = \sum_0^n f_i\,D_x^i. \qquad (5)$$

Here f_i are the coefficients of equation (3) and $a_i \in \mathcal{F}$.

ii). The following functions

$$\rho_{-1} = \frac{1}{a_1}, \qquad \rho_0 = \frac{a_0}{a_1}, \qquad \rho_i = \operatorname{res} L^i, \qquad i \in \mathbb{N} \qquad (6)$$

are conserved densities for equation (3).

The conservation laws with densities (6) are called *canonical*.

iii). If equation (3) possesses an infinite series of conserved densities, then there exist the formal Lax operator L and a series S of the form

$$S = s_1\, D_x + s_0\, + s_{-1}\, D_x^{-1} + s_{-2}\, D^{-2} + \cdots\,,$$

such that

$$S_t + A^t\, S + S\, A = 0, \quad S^t = -S, \quad L^t = -S^{-1} L S,$$

where the upper index t stands for a formal conjugation.

iv). Under the conditions of item iii) densities (6) with $i = 2k$ are of the form $\rho_{2k} = D_x(\sigma_k)$ for some functions σ_k.

Idea of the proof.

i). Rewrite equation (3) and its commuting flow (4) in the form

$$\boldsymbol{u}_t = A(\boldsymbol{u}), \qquad \boldsymbol{u}_\tau = B(\boldsymbol{u}), \qquad \text{where} \quad B = \sum_0^m g_i\, D_x^i. \qquad (7)$$

Compatibility of (7) leads to the operator identity

$$B_t - [A,\, B] = A_\tau.$$

For large m we may "ignore" the r.h.s. since it has a small order comparing with the other terms. In other words, the operator B satisfies $L_t = [A,\, L]$ approximately. Then the series of first order $L_m = B^{1/m}$ is an approximate solution as well. The gluing of the first order approximate solutions corresponding to different commuting flows into an exact formal Lax operator L is similar to the scalar case.

ii). It follows from known Adler's theorem.

Hamiltonian and recursion operators.

To define the canonical conserved densities we have considered differential operators and series with scalar coefficients from $\mathcal{F}$. However it is not enough for Hamiltonian and recursion operators and we should extend the ring of coefficients.

Denote by $R_{i,j}$ a $\mathcal{F}$-linear operator that acts on vectors by the rule

$$R_{i,j}(\boldsymbol{v}) = \boldsymbol{u}_i\,(\boldsymbol{u}_j, \boldsymbol{v}).$$

It is easy to see that

$$R_{i,j}R_{p,q} = (\boldsymbol{u}_j, \boldsymbol{u}_p)R_{i,q}, \qquad R_{i,j}^T = R_{j,i}, \qquad \operatorname{trace} R_{i,j} = (\boldsymbol{u}_i, \boldsymbol{u}_j),$$

$$D_x \circ R_{i,j} = R_{i,j}D_x + R_{i+1,j} + R_{i,j+1}.$$

Denote by $\mathcal{O}$ the algebra over $\mathcal{F}$, generated by operators $R_{i,j}$ and by the unity operator.

The Frechet derivatives of elements from $\mathcal{F}$ are differential operators with coefficients from $\mathcal{O}$. For instance, the Frechet derivative of the r.h.s. F of an equation

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + f_2 \boldsymbol{u}_{xx} + f_1 \boldsymbol{u}_x + f_0 \boldsymbol{u} \tag{8}$$

equals

$$F_* = D_x^3 + \sum_k f_k D_x^k + \sum_{i,j,k} \frac{\partial f_k}{\partial u_{[i,j]}} \left(R_{k,i}\, D_x^j + R_{k,j}\, D_x^i \right), \tag{9}$$

where $i, j, k = 0, \ldots, 2$. We will call such differential operators local.

Notice that the operator A in (5) is not a Frechet derivative of F as in the scalar case!

Example 5. The local operator

$$\mathcal{H} = u_{[0,0]}^2\, D_x + D_x \circ u_{[0,0]}^2 + 2\, u_{[0,0]}\, (R_{0,1} - R_{1,0})$$

is Hamiltonian one.

Any local Hamiltonian operator of order 1 has the form

$$\mathcal{H} = A\, D_x + D_x \circ A + s_4(R_{1,0} - R_{0,1}) + s_5(R_{2,0} - R_{0,2}) + s_6(R_{2,1} - R_{1,2}), \tag{10}$$

where

$$A = s_0 + s_1 R_{0,0} + s_2(R_{0,1} + R_{1,0}) + s_3 R_{1,1},$$

$s_i \in \mathcal{F}$.

Proposition. Suppose that the coefficients of operator (10) depend on $u_{[0,0]}, \ldots, u_{[2,2]}$ and $s_0 \neq 0$. This operator is Hamiltonian iff the coefficients have the following form :

$$s_1 = u_{[0,1]}^2 (s_0\, u_{[0,0]}\, \psi)^{-2} \left(u_{[0,1]} \frac{\partial s_0}{\partial u_{[0,1]}} + 2\, s_0 \right)^2 - \frac{s_0}{u_{[0,0]}},$$

$$s_2 = -u_{[0,1]}^2 (s_0\,\psi)^{-2}\,(u_{[0,0]})^{-1}\,\frac{\partial s_0}{\partial u_{[0,1]}}\left(u_{[0,1]}\,\frac{\partial s_0}{\partial u_{[0,1]}} + 2\,s_0\right),$$

$$s_3 = u_{[0,1]}^2 (s_0\,\psi)^{-2}\left(\frac{\partial s_0}{\partial u_{[0,1]}}\right)^2, \qquad s_5 = -s_2, \qquad s_6 = -s_3,$$

$$s_4 = \frac{u_{[0,1]}^2}{(s_0\,u_{[0,0]}\,\psi)^2}\left(u_{[0,1]}\,\frac{\partial s_0}{\partial u_{[0,1]}} + 2\,s_0\right)\times$$

$$\left(u_{[0,1]}\,\frac{\partial s_0}{\partial u_{[0,1]}} - 4\,s_0 + 2\,u_{[0,0]}\,\frac{\partial s_0}{\partial u_{[0,0]}}\right) + \frac{6\,u_{[0,1]}\sqrt{s_0}}{\psi\,(u_{[0,0]})^{3/2}} - \frac{s_0}{u_{[0,0]}},$$

where

$$\psi = D_x\left(\frac{(u_{[0,0]})^{1/2}}{s_0^{3/2}}\left(u_{[0,1]}\,\frac{\partial s_0}{\partial u_{[0,1]}} + 2\,s_0\right)\right).$$

Here $s_0(u_{[0,0]},\, u_{[0,1]})$ is an arbitrary function. □

If $s_0 = s_0(u_{[0,0]})$, then $s_2 = s_3 = s_5 = s_6 = 0$,

$$s_1 = -\frac{s_0 s_0'(u_{[0,0]} s_0' - 2\, s_0)}{(u_{[0,0]} s_0' - s_0)^2}, \qquad s_4 = -\frac{u_{[0,0]} s_0 (s_0')^2}{(u_{[0,0]} s_0' - s_0)^2}.$$

In the case $s_0 = \frac{1}{2}$ we get $\mathcal{H}_1 = D_x$; if $s_0 = \frac{1}{2}\, u_{[0,0]}^2$, we obtain the operator from Example 5.

Possibly the most Hamiltonian structures for vector integrable equations are non-local. For example, the Hamiltonian operator $\mathcal{H}$ and the symplectic operator $\mathcal{T}$ for the vector MKdV-equation

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + (\boldsymbol{u}, \boldsymbol{u})\, \boldsymbol{u}_x$$

are given by

$$\mathcal{H}(\boldsymbol{w}) = D_x \boldsymbol{w} + (\boldsymbol{u}, D_x^{-1} \circ \boldsymbol{u}) \boldsymbol{w} - (\boldsymbol{u}, D_x^{-1} \circ \boldsymbol{w}) \boldsymbol{u},$$

$$\mathcal{T}(\boldsymbol{w}) = D_x \boldsymbol{w} + \boldsymbol{u} D_x^{-1} \circ (\boldsymbol{u},\, \boldsymbol{w}).$$

Equations of KdV-type.

Integrable vector equations of the form

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + f_2 \boldsymbol{u}_{xx} + f_1 \boldsymbol{u}_x + f_0 \boldsymbol{u} \tag{11}$$

were studied by A.Meshkov and VS.

The coefficients f_i of the equation are scalar functions in the following six independent variables:

$$(\boldsymbol{u},\, \boldsymbol{u}),\ (\boldsymbol{u}, \boldsymbol{u}_x),\ (\boldsymbol{u}_x, \boldsymbol{u}_x),\ (\boldsymbol{u}, \boldsymbol{u}_{xx}),\ (\boldsymbol{u}_x, \boldsymbol{u}_{xx}),\ (\boldsymbol{u}_{xx}, \boldsymbol{u}_{xx}). \tag{12}$$

Several first canonical densities are given by

$$\rho_0 = -\frac{1}{3}\, f_2, \tag{13}$$

$$\rho_1 = \frac{1}{9}\, f_2^2 - \frac{1}{3}\, f_1 + \frac{1}{3}\, D_x(f_2).$$

$$\rho_2 = \frac{1}{3}\,\theta_0 - \frac{1}{3}\, f_0 - \frac{2}{81}\, f_2^3 + \frac{1}{9}\, f_1\, f_2 - D_x(\rho_1) - \frac{1}{3} D_x^2(\rho_0),$$

$$\rho_3 = \frac{1}{3}\,\theta_1 - \frac{d}{dx}\rho_2 - \frac{1}{3}\, D_x^2(\rho_1).$$

Formula (13) means that

$$-\frac{1}{3}D_t(f_2) = D_x(\theta_0)$$

for some $\theta \in \mathcal{F}$. Applying the Euler operator

$$\frac{\delta}{\delta \boldsymbol{u}} = \sum_{i \leqslant j} (-D_x)^i \, \boldsymbol{u}_j \left(\frac{\partial}{\partial u_{[i,j]}} \right) + (-D_x)^j \, \boldsymbol{u}_i \left(\frac{\partial}{\partial u_{[i,j]}} \right) \tag{14}$$

to the both sides of the conservation law, we get

$$\frac{\delta}{\delta \boldsymbol{u}} \frac{d}{dt} f_2 = -6 \, \boldsymbol{u}_6 \, \frac{d}{dx} \frac{\partial f_2}{\partial u_{[2,2]}} + \ldots,$$

where the dots mean terms with $\boldsymbol{u}_i, \; i < 4$. Hence $f_2 = \mathrm{const} \, u_{[2,2]} + \cdots$.

If equation has infinitly many conservation laws, then ρ_0 is trivial i.e. f_2 is a total x-derivative. It is convenient to rewrite such equation as

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - \frac{3}{2}\,\frac{d \ln f}{dx}\,\boldsymbol{u}_{xx} + f_1 \boldsymbol{u}_x + f_0 \boldsymbol{u}, \tag{15}$$

где $\operatorname{ord} f \leqslant 1$, i.e. $f = f(u_{[0,0]}, u_{[0,1]}, u_{[1,1]})$.

Proposition. For equation (15) that satisfies the second integrability condition, the coefficient f_1 has the following form

$$f_1 = c_1 \frac{u_{[2,2]}}{f} + h_1 u_{[1,2]}^2 + h_2 u_{[0,2]}^2 + h_3 u_{[1,2]} u_{[0,2]} + h_4 u_{[1,2]} + h_5 u_{[0,2]} + h_6.$$

Here $\operatorname{ord} h_i \leqslant 1$, and c_1 is a constant. □

The whole classification problem is not solved yet and only several special classes have been investigated completely.

Shift-invariant equations.

Consider the vector equation in the divergent form

$$\boldsymbol{u}_t = D_x \left(\boldsymbol{u}_{xx} + g_1 \boldsymbol{u}_x + g_0 \boldsymbol{u} \right).$$

The substitution $\boldsymbol{u} \to \boldsymbol{u}_x$ brings the the potential form of the equation:

$$\mathbf{u}_t = \mathbf{u}_{xxx} + f_2 \mathbf{u}_{xx} + f_1 \mathbf{u}_x,$$

where f_i depend on $(\mathbf{u}_x, \mathbf{u}_x)$, $(\mathbf{u}_x, \mathbf{u}_{xx})$, $(\mathbf{u}_{xx}, \mathbf{u}_{xx})$ only. It is clear that such equations are invariant w.r.t. translations $\boldsymbol{u} \to \boldsymbol{u} + \boldsymbol{c}$. These equations are completely classified and the list of the obtained equations is presented on the next slide:

List 1:

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + \frac{3}{2}\left(\frac{a^2\, {u_{[1,2]}}^2}{1 + a\, u_{[1,1]}} - a\, u_{[2,2]}\right)\boldsymbol{u}_x,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_{xx} + \frac{3}{2}\,\frac{u_{[2,2]}}{u_{[1,1]}}\,\boldsymbol{u}_x,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_{xx} + \frac{3}{2}\,\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{{u_{[1,2]}}^2}{{u_{[1,1]}}^2(1 + a\, u_{[1,1]})}\right)\boldsymbol{u}_x,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - \frac{3}{2}\,(p+1)\,\frac{u_{[1,2]}}{p\, u_{[1,1]}}\,\boldsymbol{u}_{xx} + \frac{3}{2}\,(p+1)\,\left(\frac{u_{[2,2]}}{u_{[1,1]}} - \frac{a\, {u_{[1,2]}}^2}{p^2\, u_{[1,1]}}\right)\boldsymbol{u}_x,$$

Here a is a constant and $p = \sqrt{1 + a\, u_{[1,1]}}$. Notice that if $a = 0$ the latter equation is reduced to

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_{xx} + 3\,\frac{u_{[2,2]}}{u_{[1,1]}}\,\boldsymbol{u}_x,$$

Auto-Bäcklund transformations.

The auto-Bäcklund transformations of the first order is defined by

$$\boldsymbol{u}_x = h\,\boldsymbol{v}_x + f\,\boldsymbol{u} + g\,\boldsymbol{v}, \tag{16}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions of the same vector equation. The functions f, g and h are (scalar) functions in variables

$$u_{[0,0]} = (\boldsymbol{u}, \boldsymbol{u}), \qquad v_{[i,j]} \stackrel{def}{=} (\boldsymbol{v}_i, \boldsymbol{v}_j), \qquad w_i \stackrel{def}{=} (\boldsymbol{u},\, \boldsymbol{v}_i), \qquad i, j \geqslant 0.$$

Example. The Bäcklund transformation for the vector Swartz-KdV equation

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_{xx} + \frac{3}{2}\,\frac{u_{[2,2]}}{u_{[1,1]}}\,\boldsymbol{u}_x$$

is given by

$$\boldsymbol{u}_x = \frac{2\,\mu}{\boldsymbol{v}_x^2}\,(\boldsymbol{u} - \boldsymbol{v},\ \boldsymbol{v}_x)\,(\boldsymbol{u} - \boldsymbol{v}) - \frac{\mu}{\boldsymbol{v}_x^2}\,(\boldsymbol{u} - \boldsymbol{v})^2\,\boldsymbol{v}_x,$$

where μ is an arbitrary parameter.

The superposition formula

$$\boldsymbol{z} = \boldsymbol{u} + (\mu - \nu)\,\frac{\nu\,(\boldsymbol{u} - \boldsymbol{v}')^2\,(\boldsymbol{u} - \boldsymbol{v}) - \mu\,(\boldsymbol{u} - \boldsymbol{v})^2\,(\boldsymbol{u} - \boldsymbol{v}')}{\big(\mu\,(\boldsymbol{u} - \boldsymbol{v}) - \nu\,(\boldsymbol{u} - \boldsymbol{v}')\big)^2},$$

corresponding to this auto-Bäcklund transformation connects 4 different solutions

$$\begin{array}{ccc} \boldsymbol{v}' & \xrightarrow{\ \mu\ } & \boldsymbol{z} \\ {\scriptstyle\nu}\Big\uparrow & & \Big\uparrow{\scriptstyle\nu} \\ \boldsymbol{u} & \xrightarrow[\ \mu\ ]{} & \boldsymbol{v} \end{array}$$

of the same equation. It defines a known integrable vector discrete model.

Equations on the sphere.

The condition $\boldsymbol{u}^2 = 1$ reduces the number of independent scalar product. Indeed, diffentiating $\boldsymbol{u}^2 = 1$, we obtain

$$(\boldsymbol{u},\, \boldsymbol{u}_x) = 0, \qquad (\boldsymbol{u},\, \boldsymbol{u}_{xx}) = -(\boldsymbol{u}_x,\, \boldsymbol{u}_x)$$

and so on. Moreover, the relation $(\boldsymbol{u},\, \boldsymbol{u}_t) = 0$ specifies f_0. So we consider equations

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + f_2 \boldsymbol{u}_{xx} + f_1 \boldsymbol{u}_x + \big((\boldsymbol{u}_x,\, \boldsymbol{u}_x) f_2 + 3(\boldsymbol{u}_x,\, \boldsymbol{u}_{xx})\big)\boldsymbol{u}, \quad (17)$$

where $\quad f_i = f_i\big((\boldsymbol{u}_x,\, \boldsymbol{u}_x), (\boldsymbol{u}_x,\, \boldsymbol{u}_{xx}),\, (\boldsymbol{u}_{xx},\, \boldsymbol{u}_{xx})\big).$

List 2:

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_{xx} + \frac{3}{2}\frac{u_{[2,2]}}{u_{[1,1]}}\,\boldsymbol{u}_x,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_{xx} + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2\,(1 + a\,u_{[1,1]})}\right)\boldsymbol{u}_x,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + \frac{3}{2}\left(\frac{a^2\,u_{[1,2]}^2}{1 + a\,u_{[1,1]}} - a\,(u_{[2,2]} - u_{[1,1]}^2) + u_{[1,1]}\right)\boldsymbol{u}_x + 3\,u_{[1,2]}\,\boldsymbol{u},$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\,\frac{(p+1)\;u_{[1,2]}}{2\,p\,u_{[1,1]}}\,\boldsymbol{u}_{xx} + 3\,\frac{(p-1)\;u_{[1,2]}}{2\,p}\,\boldsymbol{u}$$

$$+\frac{3}{2}\left(\frac{(p+1)\,u_{[2,2]}}{u_{[1,1]}} - \frac{(p+1)\,a\,{u_{[1,2]}}^2}{p^2 u_{[1,1]}} + u_{[1,1]}\,(1-p)\right)\boldsymbol{u}_x,$$

where $p = \pm\sqrt{1 + a u_{[1,1]}}$ and a is a parameter.

Anisotropic equations

Example. Consider equation (2):

$$\boldsymbol{u}_t = \left(\boldsymbol{u}_{xx} + \frac{3}{2}(\boldsymbol{u}_x,\ \boldsymbol{u}_x)\boldsymbol{u}\right)_x + \frac{3}{2}\langle \boldsymbol{u}, \boldsymbol{u}\rangle\, \boldsymbol{u}_x, \qquad \boldsymbol{u}^2 = 1.$$

Here $\langle \boldsymbol{a},\ \boldsymbol{b}\rangle = (\boldsymbol{a},\ R\,\boldsymbol{b})$, where $R = \mathrm{diag}(r_1, ..., r_N)$ is arbitrary constant matrix.

This equation has a Lax representation $L_t = [A, L]$, where

$$L = D_x + \begin{pmatrix} 0 & \Lambda \mathbf{u} \\ \mathbf{u}^T \Lambda, & 0 \end{pmatrix}.$$

Here

$$\Lambda = \frac{1}{\lambda}\mathrm{diag}\,(\sqrt{1-\lambda^2 r_1}, \ldots, \sqrt{1-\lambda^2 r_N}).$$

It was a first explicit example of a Lax operator with the spectral parameter lying on the algebraic curve

$$\lambda_1^2 + r_1 = \lambda_2^2 + r_2 - \cdots = \lambda_N^2 + r_N$$

of genus $1 + (N-3)2^{N-2}$.

If $N = 3$, then (2) is a commuting flow of the Landau - Lifshitz equation.

Besides, (2) defines a commuting flow for the Noemann system

$$\boldsymbol{u}_{xx} = -\Big((\boldsymbol{u}_x,\, \boldsymbol{u}_x) + (\boldsymbol{u},\, R\,\boldsymbol{u})\Big)\,\boldsymbol{u} + R\,\boldsymbol{u}, \qquad \boldsymbol{u}^2 = 1,$$

describing the dynamics of a particle on the sphere under the quadratic potential $\mathcal{U} = \frac{1}{2}(\boldsymbol{u},\, R\,\boldsymbol{u})$. More precisely, if we eliminate the derivatives $\boldsymbol{u}_{xx}$ and $\boldsymbol{u}_{xxx}$ from (2), then the reduced system

$$\boldsymbol{u}_t = \frac{1}{2}\Big((\boldsymbol{u}_x,\, \boldsymbol{u}_x) + (\boldsymbol{u},\, R\,\boldsymbol{u})\Big)\,\boldsymbol{u}_x - (\boldsymbol{u}_x,\, R\,\boldsymbol{u})\,\boldsymbol{u} + R\,\boldsymbol{u}_x$$

is a commuting flow for the Noemann system. □

In this *anisotropic* example the coefficients of vector equation (3) depend on two different independent scalar products $(\cdot\,,\cdot)$ and $\langle\cdot\,,\cdot\rangle$.

All anisotropic equations

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + f_2\boldsymbol{u}_{xx} + f_1\boldsymbol{u}_x + f_0\boldsymbol{u},$$

on the sphere $u_{[0,0]} = 1$ have been found (21 eqs.). In this case the coefficients f_i depend on both isotropic variables

$$u_{[1,1]},\ u_{[1,2]},\ u_{[2,2]}$$

and anisotropic one

$$v_{[0,0]},\ v_{[0,1]},\ v_{[1,1]},\ v_{[0,2]},\ v_{[1,2]},\ v_{[2,2]},$$

where

$$v_{[i,j]} = (\boldsymbol{u}_i,\ R(\boldsymbol{u}_j)).$$

List 3:

All "‘rational"’ equations of the list are:

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + \left(\frac{3}{2}\, u_{[1,1]} + v_{[0,0]}\right) \boldsymbol{u}_1 + 3\, u_{[1,2]}\, \boldsymbol{u}_0,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\,\frac{u_{[1,2]}}{u_{[1,1]}}\,\boldsymbol{u}_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2} + \frac{v_{[1,1]}}{u_{[1,1]}}\right)\boldsymbol{u}_1,$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\boldsymbol{u}_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2} - \frac{(v_{[0,1]} + u_{[1,2]})^2}{q\, u_{[1,1]}} + \frac{v_{[1,1]}}{u_{[1,1]}}\right)\boldsymbol{u}_1,$$

where $q = u_{[1,1]} + v_{[0,0]} + a$.

$$\boldsymbol{u}_t=\boldsymbol{u}_3-3\frac{v_{[0,1]}}{v_{[0,0]}}\boldsymbol{u}_2-3\left(\frac{v_{[0,2]}}{v_{[0,0]}}-2\frac{v_{[0,1]}^2}{v_{[0,0]}^2}\right)\boldsymbol{u}_1+3\left(u_{[1,2]}-\frac{v_{[0,1]}}{v_{[0,0]}}u_{[1,1]}\right)\boldsymbol{u},$$

$$\boldsymbol{u}_t=\boldsymbol{u}_3-3\frac{v_{[0,1]}}{v_{[0,0]}}\boldsymbol{u}_2-3\left(\frac{2v_{[0,2]}+v_{[1,1]}+a}{2v_{[0,0]}}-\frac{5}{2}\frac{v_{[0,1]}^2}{v_{[0,0]}^2}\right)\boldsymbol{u}_1+$$

$$+3\left(u_{[1,2]}-\frac{v_{[0,1]}}{v_{[0,0]}}u_{[1,1]}\right)\boldsymbol{u},$$

$$\boldsymbol{u}_t=\boldsymbol{u}_3-3\frac{v_{[0,1]}}{v_{[0,0]}}\left(\boldsymbol{u}_2+u_{[1,1]}\boldsymbol{u}\right)+3u_{[1,2]}\boldsymbol{u}+$$

$$+\frac{3}{2}\Big(-\frac{u_{[2,2]}}{v_{[0,0]}}+\frac{(u_{[1,2]}+v_{[0,1]})^2}{v_{[0,0]}(v_{[0,0]}+u_{[1,1]})}+$$

$$+\frac{(v_{[0,0]}+u_{[1,1]})^2}{v_{[0,0]}}+\frac{v_{[0,1]}^2-v_{[0,0]}\,v_{[1,1]}}{v_{[0,0]}^2}\Big)\boldsymbol{u}_1.$$

Last results.

1. Hyperbolic equations

For hyperbolic equations

$$u_{xy} = \Psi(u, u_x, u_y)$$

the symmetry approach assumes the existence of both x-symmetries of the form

$$u_t = A(u, u_x, u_{xx}, \dots),$$

and y-symmetries of the form

$$u_\tau = B(u, u_y, u_{yy}, \dots).$$

For example, the famous integrable sin-Gordon equation

$$u_{xy} = \sin u$$

admits the symmetries

$$u_t = u_{xxx} + \frac{1}{2}u_x^3, \qquad u_\tau = u_{yyy} + \frac{1}{2}u_y^3.$$

We consider equations

$$\boldsymbol{u}_{xy} = h_0\boldsymbol{u} + h_1\boldsymbol{u}_x + h_2\boldsymbol{u}_y$$

on the sphere. Here h_i are some scalar-valued functions depending on two different scalar products $(\cdot\,,\cdot)$ and $\langle\cdot\,,\cdot\rangle$ between vectors $\boldsymbol{u}, \boldsymbol{u}_x$ and $\boldsymbol{u}_y$. The sphere is defined by the equation $\boldsymbol{u}^2 = 1$.

Example. A hyperbolic integrable equation on the sphere is given by

$$\boldsymbol{u}_{xy} = \frac{\boldsymbol{u}_x}{\langle\boldsymbol{u},\boldsymbol{u}\rangle}\Big(\langle\boldsymbol{u},\boldsymbol{u}_y\rangle + \sqrt{1+\langle\boldsymbol{u},\boldsymbol{u}\rangle(\boldsymbol{u}_x,\boldsymbol{u}_x)^{-2}}\;\varphi\Big) - (\boldsymbol{u}_x,\boldsymbol{u}_y)\boldsymbol{u},$$

$$\text{where} \qquad \varphi = \sqrt{\langle\boldsymbol{u},\boldsymbol{u}_y\rangle^2 + \langle\boldsymbol{u},\boldsymbol{u}\rangle(1-\langle\boldsymbol{u}_y,\boldsymbol{u}_y\rangle)}.$$

In the case $N = 2$ this equation is equivalent to

$$u_{xy} = \operatorname{sn}(u)\sqrt{u_x^2+1}\sqrt{u_y^2+1}.$$

2. Integrability conditions

Consider equations

$$\boldsymbol{u}_t = a^{-3}\boldsymbol{u}_{xxx} + f_2\boldsymbol{u}_{xx} + f_1\boldsymbol{u}_x + f_0\boldsymbol{u}. \tag{18}$$

The recurrent formula for densities of canonical conservation laws

$$D_t\,\rho_i = D_x\,\theta_i, \qquad i = -1, 0, 1, 2, \dots$$

is given by

$$\rho_{n+2} = \frac{a}{3}\left(\theta_n - f_0\,\delta_{n,0} - 2\,af_2\,\rho_{n+1} - f_2\,\frac{d}{dx}\rho_n - f_1\,\rho_n\right)$$
$$- \frac{a}{3}\,f_2 \sum_{i+j=n}\rho_i\,\rho_j - \frac{1}{3}\,a^{-2}\sum_{i+j+k=n}\rho_i\,\rho_j\,\rho_k - a^{-1}\sum_{i+j=n+1}\rho_i\,\rho_j$$
$$- a^{-2}\frac{d}{dx}\,a\,\rho_{n+1} - \frac{1}{2}\,a^{-2}\frac{d}{dx}\sum_{i+j=n}\rho_i\,\rho_j - \frac{1}{3}\,a^{-2}\,\frac{d^2}{dx^2}\,\rho_n, \qquad n \geqslant 0, \tag{19}$$

where $\delta_{i,j}$ is the Kronecker delta, and

$$\rho_{-1} = a,$$

$$\rho_0 = -\frac{1}{3}\,a^3 f_2 - \frac{d}{dx}\ln a,$$

$$\rho_1 = \frac{a}{3}\theta_{-1} + a^{-1}\rho_0^2 - \frac{1}{3}\,a^2 f_1 + a^{-3}\left(\frac{d}{dx}\,a\right)^2 +$$

$$2\,a^{-2}\rho_0\frac{d}{dx}\,a - a^{-1}\,\frac{d}{dx}\,\rho_0 - \frac{1}{3}\,a^{-2}\frac{d^2}{dx^2}\,a.$$

Notice that in the expressions for ρ_i we use besides the coefficients f_i of equation (18) the fluxes θ_j of the canonical conservation laws with $j \leqslant i-2$.

Example. An integrable vector Harry Dym equation

$$\boldsymbol{u}_t = (\mathbf{u},\mathbf{u})^{3/2}\boldsymbol{u}_{xxx}.$$

Formula (19) are universal. It is valid for the equations with separant 1 and with a fuctional separant. It is valid for anisotropic equations as well. Moreover, it is hold even for equations with the constant vector.

The vector KdV equation

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + (\boldsymbol{c}, \boldsymbol{u})\, \boldsymbol{u}_x + (\boldsymbol{c}, \boldsymbol{u}_x)\, \boldsymbol{u} - (\boldsymbol{u}, \boldsymbol{u}_x)\, \boldsymbol{c},$$

belongs to the class of equations of the form

$$\boldsymbol{u}_t = f_3 \boldsymbol{u}_{xxx} + f_2 \boldsymbol{u}_{xx} + f_1 \boldsymbol{u}_x + f_0 \boldsymbol{u} + h\, \boldsymbol{c}.$$

In this case the coefficients depend on

$$(\boldsymbol{u}_i,\, \boldsymbol{u}_j), \quad \text{and} \quad (\boldsymbol{u}_i,\, \boldsymbol{c}), \qquad i \leqslant j. \tag{20}$$

Without loss of generality we may assume that $(\boldsymbol{c}, \boldsymbol{c}) = 1$.

Consider equations of the form

$$\boldsymbol{u}_t = A(\boldsymbol{u}) + h\,\boldsymbol{c}, \tag{21}$$

where $\boldsymbol{c}$ is a constant vector and A is a linear differential operator with coefficients depending on variables (20).

Theorem. Integrable equations possess a formal series

$$L = a_1\, D_x + a_0\, + a_{-1}\, D_x^{-1} + \cdots$$

with scalar coefficients such that

$$L_t = [A,\, L].$$

Corollary. For the equations of the form

$$\boldsymbol{u}_t = a^{-3}\boldsymbol{u}_{xxx} + f_2\boldsymbol{u}_{xx} + f_1\boldsymbol{u}_x + f_0\boldsymbol{u} + h\,\boldsymbol{c}$$

formula (19) for canonical densities are valid.