

Zeros of polynomials and solvable nonlinear evolution equations

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Abstract

There exist convenient explicit formulas expressing the time-derivative (of any order k) of each of the N zeros of a (monic) time-dependent polynomial of degree N in terms of: (i) the time-derivatives of the same order k of the N coefficients of that polynomial; and (ii) the N zeros of that polynomial and their time-derivatives of order less than k . This allows the identification of many new dynamical systems *solvable/integrable* by algebraic operations including N -body problems characterized by Newtonian (accelerations equal forces) equations of motion describing N , *nonlinearly* interacting, point-particles moving in the *complex* plane, or equivalently in the *real* Cartesian plane.

The same approach can also be used to identify *new* solvable systems of *nonlinear PDEs*; as well as *new differential algorithms* to compute *all* the zeros of generic polynomials of *arbitrary* degree N . This approach moreover yields infinite hierarchies of all these *solvable nonlinear* evolution equations, via the introduction of the notion---of mathematical interest in its own right---of generations of monic polynomials, all of the same arbitrary degree N , with the N coefficients of the polynomials of the next generation coinciding with every one of the $N!$ permutations of the N zeros of each polynomial of the current generation. In this talk I will describe these recent developments, which have been reported in several papers published in the last 1-2 years---some authored by myself alone and some co-authored with **Oksana Bihun**, with **Mario Bruschi** or with **Francois Leyvraz**---which are reviewed in a 170-page book I just completed (see Bibliography below).

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Coefficients and zeros of monic polynomials

$$p_N(z; \vec{y}; \tilde{x}) = z^N + \sum_{m=1}^N (y_m z^{N-m})$$

$$p_N(z; \vec{y}; \tilde{x}) = \prod_{n=1}^N (z - x_n)$$

\vec{y} is an N -vector: its N components y_m are the N coefficients of the polynomial $p_N(z; \vec{y}; \tilde{x})$ of degree N in the independent (*complex*) variable z .

\tilde{x} is an **unordered** set of N numbers x_n , which are the N zeros of the polynomial $p_N(z; \vec{y}; \tilde{x})$. We denote as $\vec{x}_{[\mu]}$ the N -vector, the N components $x_{[\mu],n}$ of which are given by the specific permutation--- labeled by the integer index μ in the range $1 \leq \mu \leq N!$ ---of the N numbers x_n . Hence, to any given \tilde{x} , there generally correspond $N!$ *different* N -vectors $\vec{x}_{[\mu]}$. We will come back to this notion below.

Time-dependent monic polynomials

$$\psi_N \left(\zeta; \vec{\gamma}(t); \tilde{\xi}(t) \right) = \zeta^N + \sum_{m=1}^N [\gamma_m(t) \zeta^{N-m}]$$

$$\psi_N \left(\zeta; \vec{\gamma}(t); \tilde{\xi}(t) \right) = \prod_{n=1}^N [\zeta - \xi_n(t)]$$

$\vec{\gamma}(t)$ is a time-dependent N -vector: its N components $\gamma_m(t)$ are the N *coefficients* of the time-dependent polynomial $\psi_N \left(\zeta; \vec{\gamma}(t); \tilde{\xi}(t) \right)$ of degree N in the independent (generally *complex*) variable ζ . $\tilde{\xi}(t)$ is an **unordered** set of N numbers $\xi_n(t)$, which are the N *zeros* of the polynomial $\psi_N \left(\zeta; \vec{\gamma}(t); \tilde{\xi}(t) \right)$ (but they get *ordered* via their dependence on time if this dependence is *continuous*).

Two very useful formulas [1]

$$\dot{\xi}_n(t) = - \left\{ \prod_{j=1, j \neq n}^N [\xi_n(t) - \xi_j(t)] \right\}^{-1} \sum_{m=1}^N \{ \dot{\gamma}_m(t) [\xi_n(t)]^{N-m} \},$$

$$\ddot{\xi}_n(t) = \sum_{j=1, j \neq n}^N \left[\frac{2 \dot{\xi}_n(t) \dot{\xi}_j(t)}{\xi_n(t) - \xi_j(t)} \right]$$

$$- \left\{ \prod_{j=1, j \neq n}^N [\xi_n(t) - \xi_j(t)] \right\}^{-1} \sum_{m=1}^N \{ \ddot{\gamma}_m(t) [\xi_n(t)]^{N-m} \},$$

$$\gamma_m(t) = (-1)^m \sigma_m[\tilde{\xi}(t)].$$

The second-derivative formula is particularly useful and is referred below as (***) .

Extensions

These formulas have been extended to the third and fourth derivatives in [3], to derivatives of *arbitrary* order in [7] and to the case of *discrete time* in [15]. These results are not reported nor discussed in this talk, but they are instrumental to demonstrate the *solvable* character of some of the dynamical systems described in the 18 References (see **Bibliography** above).

An application of the first formula

$$\dot{\xi}_n(t) = - \left\{ \prod_{j=1, j \neq n}^N [\xi_n(t) - \xi_j(t)] \right\}^{-1} \sum_{m=1}^N \{ \dot{\gamma}_m(t) [\xi_n(t)]^{N-m} \},$$

$$\gamma_m(t) = \gamma_m(0) + [f_m(t) - f_m(0)] [c_m - \gamma_m(0)],$$

$$f_m(T) - f_m(0) = 1, \quad \gamma_m(T) = c_m;$$

$$\dot{\gamma}_m(t) = \dot{f}_m(t) [c_m - \gamma_m(0)]$$

[Special case: $f_m(t) = t, \dot{f}_m(t) = 1, T = 1.$]

Nonlinear differential algorithm to compute all the zeros of a generic polynomial [13,14]

$$P_N(z; \vec{c}; \tilde{x}) = z^N + \sum_{m=1}^N (c_m z^{N-m}) = \prod_{n=1}^N (z - x_n)$$

Now introduce the t -dependent polynomial

$$\begin{aligned} p_N(z; \vec{\gamma}(t); \tilde{y}(t)) &= z^N + \sum_{m=1}^N [\gamma_m(t) z^{N-m}] \\ &= \prod_{n=1}^N [z - y_n(t)] \end{aligned}$$

There holds then the following

Proposition. Consider the following system of N nonlinear first-order differential equations satisfied by the N zeros $y_n(t)$ of this polynomial:

$$\dot{y}_n(t) = - \left\{ \prod_{s=1, s \neq n}^N [y_n(t) - y_s(t)]^{-1} \right\} .$$

$$\cdot \sum_{m=1}^N \left\{ \dot{f}_m(t) [c_m - \gamma_m(0)] [y_n(t)]^{N-m} \right\} ,$$

$$f_m(T) - f_m(0) = 1,$$

$$\gamma_m(0) = (-1)^m \sigma_m(\tilde{y}(0)) , \quad m = 1, \dots, N ;$$

$$\sigma_m(\tilde{y}) = \sum_{1 \leq s_1 < s_2 < \dots < s_m \leq N} (y_{s_1} y_{s_2} \dots y_{s_m}) .$$

Then:

$$x_n = y_n(T) .$$

It is thus seen that the zeros x_n of the polynomial $P_N(z; \vec{c}; \tilde{x})$ can be computed---once the N coefficients c_m of this polynomial have been assigned---via the following procedure. *Step one:* choose (arbitrarily!) N complex numbers $y_n(0)$. *Step two:* compute the N quantities $\gamma_m(0)$. *Step three:* integrate (numerically) the above system of differential equations from $t=0$ to $t=T$, starting from the N initial data $y_n(0)$, getting thereby the N values $y_n(T)$, which give the sought result, $x_n = y_n(T)$.

[Subcase: $f_m(t) = t, \dot{f}_m(t) = 1, T = 1$.]

The algorithm to evaluate *all* the zeros of a *generic* polynomial of *arbitrary* degree N described above requires the solution of a system of N nonlinearly coupled *first-order* ODEs (starting from an *arbitrary* set of initial data). Quite recently a *novel* differential algorithm to evaluate *all* the zeros of a *generic* polynomial of *arbitrary* degree N has been identified, which requires the solution of a system of N nonlinearly coupled *second-order* ODEs. [17] While this might be considered a disadvantage with respect to the previous algorithm, it is on the other hand remarkably neater and therefore worthy of being widely known at least because of its elegance.

Let a *generic* (monic) polynomial be defined as follows in terms of its N coefficients c_m and its N zeros x_n :

$$p_N(z) = z^N + \sum_{m=1}^N (c_m z^{N-m}) = \prod_{n=1}^N (z - x_n)$$

The task is to compute the N zeros x_n (all different among themselves: "genericity" of this polynomial).

The algorithm. Consider the dynamical system characterized by the following N *second-order* ODEs (of Newtonian type: "accelerations equal forces")

$$\ddot{\xi}_n(t) = \sum_{j=1, j \neq n}^N \left[\frac{2 \dot{\xi}_n(t) \dot{\xi}_j(t)}{\xi_n(t) - \xi_j(t)} \right]$$

which characterizes the motion in the complex ξ -plane of the N points identified by the N coordinates $\xi_n(t)$; here and below superimposed dots indicate of course differentiation with respect to the real parameter t ("dimensionless time": the independent variable). This system is, in fact, an *integrable Hamiltonian* system, to which the honorary name "goldfish" has been attributed because of the neatness of its equations of motion, which are clearly *autonomous* and invariant under *any arbitrary* common *translation* or *rescaling* of the dependent variables $\xi_n(t)$.

Next, integrate this system of N *second-order* ODEs from $t=0$ to $t=1$, starting---at the "initial time" $t=0$ ---from N , *arbitrarily assigned*, generally *complex*, N *initial values* $\xi_n(0)$, with the corresponding *initial velocities* given by the following *explicit* formula in terms of these N initial data *initial values* $\xi_j(0)$ and of the N coefficients c_m of the polynomial $p_N(z)$:

$$\dot{\xi}_n(0) = - \frac{[\xi_n(0)]^N + \sum_{m=1}^N \{c_m [\xi_n(0)]^{N-m}\}}{\prod_{j=1, j \neq n}^N [\xi_n(0) - \xi_j(0)]} = - \frac{p_N[\xi_n(0)]}{\prod_{j=1, j \neq n}^N [\xi_n(0) - \xi_j(0)]}$$

There thus obtain the N zeros x_n of the polynomial $p_N(z)$:

$$\xi_n(1) = x_n , \quad p_N(x_n) = 0 .$$

New solvable dynamical systems

Many new *solvable* models---including *isochronous* ones---can be manufactured by taking advantage of the formula (***):

$$\ddot{\xi}_n(t) = \sum_{j=1, j \neq n}^N \left[\frac{2 \dot{\xi}_n(t) \dot{\xi}_j(t)}{\xi_n(t) - \xi_j(t)} \right]$$

$$- \left\{ \prod_{j=1, j \neq n}^N [\xi_n(t) - \xi_j(t)] \right\}^{-1} \sum_{m=1}^N \left\{ \ddot{\gamma}_m(t) [\xi_n(t)]^{N-m} \right\},$$

$$\gamma_m(t) = (-1)^m \sigma_m[\tilde{\xi}(t)],$$

$$\dot{\gamma}_m(t) = (-1)^m \sum_{n=1}^N \left\{ \sigma_{n,m}[\tilde{\xi}(t)] \dot{\xi}_n(t) \right\}.$$

Indeed, assume that a “solvable” dynamical system of Newtonian type (“acceleration equal force”) reads

$$\ddot{\vec{\gamma}}(t) = \vec{f}[\vec{\gamma}(t); \dot{\vec{\gamma}}(t); t]$$

or equivalently

$$\ddot{\gamma}_m(t) = f_m[\vec{\gamma}(t); \dot{\vec{\gamma}}(t); t], \quad m = 1, \dots, N.$$

Then insert these equations of motion in the right-hand side of (***) . In this manner you obtain a new “solvable” dynamical system of Newtonian type. Several new examples are exhibited in references [1-11, 18], see above. (Ref. [12] treats systems of nonlinear PDEs). Below only one *well-known* example and two *new* examples are reported.

The isochronous goldfish model

$$\ddot{\gamma}_m(t) = i \omega \dot{\gamma}_m(t) ,$$

$$\gamma_m(t) = \gamma_m(0) + \frac{i \dot{\gamma}_m(0) [1 - \exp(i\omega t)]}{\omega} ,$$

$$\ddot{\xi}_n(t) = i \omega \dot{\xi}_n(t) + \sum_{j=1, j \neq n}^N \left[\frac{2 \dot{\xi}_n(t) \dot{\xi}_j(t)}{\xi_n(t) - \xi_j(t)} \right] .$$

Solution: the N coordinates $\xi_n(t)$ are the N roots of

$$\sum_{n=1}^N \left[\frac{\dot{\xi}_n(0)}{\xi - \xi_n(0)} \right] = \frac{i \omega}{\exp(i \omega t) - 1} .$$

Isochronous with period $T = 2\pi/\omega$ ---or a, generally small (much less than $N!$), *integer* multiple of T .

The “goldfish-CM” model [2]

$$\ddot{\gamma}_m(t) = -\omega^2 \gamma_m(t) + 2 g_m^2 \sum_{j=1, j \neq m}^N [\gamma_m(t) - \gamma_j(t)]^{-3};$$

$$\ddot{\xi}_n(t) = \sum_{j=1, j \neq n}^N \left[\frac{2 \dot{\xi}_n(t) \dot{\xi}_j(t)}{\xi_n(t) - \xi_j(t)} \right]$$

$$- \left\{ \prod_{j=1, j \neq n}^N [\xi_n(t) - \xi_j(t)] \right\}^{-1} \sum_{m=1}^N \left\{ \ddot{\gamma}_m(t) [\xi_n(t)]^{N-m} \right\},$$

$$\gamma_m(t) = (-1)^m \sigma_m[\tilde{\xi}(t)].$$

Isochronous with period $T = 2\pi/\omega$ ---or a, generally small (much less than $N!$), *integer* multiple of T .

The “goldfish-CM” model equations of motion [2]

$$\ddot{\xi}_n(t) = \sum_{j=1, j \neq n}^N \left[\frac{2 \dot{\xi}_n(t) \dot{\xi}_j(t)}{\xi_n(t) - \xi_j(t)} \right] - \left\{ \prod_{j=1, j \neq n}^N [\xi_n(t) - \xi_j(t)] \right\}^{-1} \cdot$$

$$\cdot \sum_{m=1}^N \left\{ \left\{ -\omega^2 \gamma_m(t) + 2 g_m^2 \sum_{j=1, j \neq m}^N [\gamma_m(t) - \gamma_j(t)]^{-3} \right\} [\xi_n(t)]^{N-m} \right\},$$

$$\gamma_m(t) = (-1)^m \sigma_m[\tilde{\xi}(t)],$$

$$\sigma_m(\tilde{\xi}) = \sum_{1 \leq s_1 < s_2 < \dots < s_m \leq N} \left(\xi_{s_1} \xi_{s_2} \dots \xi_{s_m} \right).$$

An *integrable Hamiltonian N*-body problem in the plane featuring *N* arbitrary functions [11]

$$\ddot{\xi}_n(t) = \sum_{j=1, j \neq n}^N \left[\frac{2 \dot{\xi}_n(t) \dot{\xi}_j(t)}{\xi_n(t) - \xi_j(t)} \right]$$

$$- \left\{ \prod_{j=1, j \neq n}^N [\xi_n(t) - \xi_j(t)] \right\}^{-1} \sum_{m=1}^N \{ g_m(\gamma_m(t)) [\xi_n(t)]^{N-m} \},$$

$$\gamma_m(t) = (-1)^m \sigma_m[\tilde{\xi}(t)],$$

$$\sigma_m(\tilde{\xi}) = \sum_{1 \leq s_1 < s_2 < \dots < s_m \leq N} (\xi_{s_1} \xi_{s_2} \dots \xi_{s_m}).$$

This finding is an obvious consequence of the fact that the system of ***decoupled*** ODEs

$$\ddot{\gamma}_m(t) = g_m(\gamma_m(t))$$

is clearly both ***Hamiltonian*** and ***integrable***; and of course both these properties are inherited by the above system of nonlinear coupled second-order ODEs satisfied by the “particle coordinates” $\xi_n(t)$ because the transformation from the N coordinates $\gamma_m(t)$ to the N coordinates $\xi_n(t)$ is clearly a ***canonical*** transformation in the Hamiltonian context.

These examples are obtained directly from the formula (**). But the process can be iterated over and over again.

This motivated the idea to introduce and investigate **the generations of monic polynomials obtained by replacing the *coefficients* of the polynomials of the next generation with the *zeros* of a polynomial of the previous generation. [4]**

The seed polynomial

$$p_N^{(0)}(z; \vec{y}^{(0)}; \tilde{x}^{(0)}) = z^N + \sum_{m=1}^N \left[y_m^{(0)} z^{N-m} \right],$$

$$p_N^{(0)}(z; \vec{y}^{(0)}; \tilde{x}^{(0)}) = \prod_{n=1}^N \left[z - x_n^{(0)} \right].$$

The first generation of $N!$ monic polynomials

$$p_N^{(\mu_1;1)}(z; \vec{y}_{[\mu_1]}^{(1)}; \tilde{x}_{[\mu_1]}^{(1)}) = z^N + \sum_{m=1}^N \left[y_{[\mu_1],m}^{(1)} z^{N-m} \right],$$

$$p_N^{(\mu_1;1)}(z; \vec{y}_{[\mu_1]}^{(1)}; \tilde{x}_{[\mu_1]}^{(1)}) = \prod_{n=1}^N \left[z - x_{[\mu_1],n}^{(1)} \right],$$

$$\vec{y}_{[\mu_1]}^{(1)} = \tilde{x}_{[\mu_1]}^{(0)}; \quad y_{[\mu_1],m}^{(1)} = x_{[\mu_1],m}^{(0)}.$$

The *integer* index μ_1 taking values in the range $1 \leq \mu_1 \leq N!$ labels the permutations of the *a priori* unordered set $\tilde{x}^{(0)}$.

The second generation of $(N!)^2$ monic polynomials

$$p_N^{(\vec{\mu}^{(2)};2)} \left(z; \vec{y}_{[\vec{\mu}^{(2)}]}^{(2)}; \tilde{x}_{[\vec{\mu}^{(2)}]}^{(2)} \right) = z^N + \sum_{m=1}^N \left[y_{[\vec{\mu}^{(2)}],m}^{(2)} z^{N-m} \right],$$

$$p_N^{(\vec{\mu}^{(2)};2)} \left(z; \vec{y}_{[\vec{\mu}^{(2)}]}^{(2)}; \tilde{x}_{[\vec{\mu}^{(2)}]}^{(2)} \right) = \prod_{n=1}^N \left[z - x_{[\vec{\mu}^{(2)}],n}^{(2)} \right],$$

$$\vec{y}_{[\vec{\mu}^{(2)}]}^{(2)} = \vec{x}_{[\vec{\mu}^{(2)}]}^{(1)} = \vec{x}_{[\mu_1],[\mu_2]}^{(1)}; \quad y_{[\vec{\mu}^{(2)}],m}^{(2)} = x_{[\vec{\mu}^{(2)}],m}^{(1)}.$$

The 2-vector $\vec{\mu}^{(2)} = (\mu_1, \mu_2)$ has *integer* components μ_1, μ_2 taking values in the range $1 \leq \mu_1, \mu_2 \leq N!$. The label μ_2 identifies the permutation of the *a priori unordered* set $\tilde{x}_{[\mu_1]}^{(1)}$ the components of which are the N zeros of the polynomial of the previous generation with index μ_1 , i. e. the N zeros of $p_N^{(\mu_1;1)} \left(z; \vec{y}_{[\mu_1]}^{(1)}; \tilde{x}_{[\mu_1]}^{(1)} \right)$.

The k -th generation of $(N!)^k$ monic polynomials

$$p_N^{(\vec{\mu}^{(k)};k)} \left(z; \vec{y}_{[\vec{\mu}^{(k)}]}^{(k)}; \tilde{x}_{[\vec{\mu}^{(k)}]}^{(k)} \right) = z^N + \sum_{m=1}^N \left[y_{[\vec{\mu}^{(k)}],m}^{(k)} z^{N-m} \right],$$

$$p_N^{(\vec{\mu}^{(k)};k)} \left(z; \vec{y}_{[\vec{\mu}^{(k)}]}^{(k)}; \tilde{x}_{[\vec{\mu}^{(k)}]}^{(k)} \right) = \prod_{n=1}^N \left[z - x_{[\vec{\mu}^{(k)}],n}^{(k)} \right],$$

$$\vec{y}_{[\vec{\mu}^{(k)}]}^{(k)} = \tilde{x}_{[\vec{\mu}^{(k)}]}^{(k-1)} = \tilde{x}_{[\vec{\mu}^{(k-1)}],[\mu_k]}^{(k-1)}; \quad y_{[\vec{\mu}^{(k)}],m}^{(k)} = x_{[\vec{\mu}^{(k)}],m}^{(k-1)}.$$

The k -vector $\vec{\mu}^{(k)} = (\mu_1, \mu_2, \dots, \mu_k)$ has *integer* components $\mu_1, \mu_2, \dots, \mu_k$ taking values in the range $1 \leq \mu_1, \mu_2, \dots, \mu_k \leq N!$. The label μ_k identifies the permutation of the *a priori unordered* set $\tilde{x}_{[\vec{\mu}^{(k-1)}]}^{(k-1)}$

the components of which are the N zeros of the polynomial of the $(k-1)$ -th generation with indices $\mu_1, \mu_2, \dots, \mu_{k-1}$, i. e. the N zeros of

$$p_N^{(\vec{\mu}^{(k-1)};k-1)} \left(z; \vec{y}_{[\vec{\mu}^{(k-1)}]}^{(k-1)}; \tilde{x}_{[\vec{\mu}^{(k-1)}]}^{(k-1)} \right).$$

Why introduce this notion of generations of monic polynomials?

Reply: Why not? And note the possibility mentioned above to generate many *solvable* dynamical systems---including many-body problems of Newtonian type (“accelerations equal forces”)---associated with such polynomials.

Question to the audience: is this notion of *generations of polynomials* new? Has it already been investigated?

Thanks for your attention,
questions are welcome!