Quasi-polynomiality of Bousquet-Mélou–Schaeffer numbers

Boris Bychkov

Higher School of Economics, Moscow

Kezenoi-Am 2017

Work in progress with P. Dunin-Barkowski

Definition

The *degeneracy* $k(\mu)$ of a permutation μ is the minimal number of transposition in its representation as a product of transposition: $k(\mu) = |\mu| - l(\mu)$, where $l(\mu)$ is the quantity of the independent cycles in μ.

Definition

The *degeneracy* $k(\mu)$ of a permutation μ is the minimal number of transposition in its representation as a product of transposition: $k(\mu) = |\mu| - l(\mu)$, where $l(\mu)$ is the quantity of the independent cycles in μ.

Definition

Generalized Hurwitz numbers $a_{k_1,\dots,k_m,\mu}$:

$$a^{\circ}_{k_1,\dots,k_m,\mu} = \frac{1}{|\mu|!}|\{(\tau_1,\dots,\tau_m), \tau_i \in S_\mu | k(\tau_i) = k_i, \tau_m \circ \dots \circ \tau_1 \in C_\mu\}|$$

- μ is the partition of the $|\mu|$,
- $S_{|\mu|}$ is the permutation group with $|\mu|$ elements,
- C_μ is the set of all permutations with the cyclic type μ in the $S_{|\mu|}$.

Bousquet-Mélou–Schaeffer numbers

Definition

Bousquet-Mélou–Schaeffer numbers $b_{m,k;\mu}$:

$$b^{\circ}_{m,k,\mu} = \frac{1}{|\mu|!}\left|\{(\tau_1,\ldots,\tau_m), \tau_i \in S_{|\mu|} \mid \sum_{i=1}^{m} k(\tau_i) = k, \tau_m \circ \ldots \circ \tau_1 \in C_\mu\}\right|,$$

where $k(\tau_i) = |\tau_i| - l(\tau_i)$ is the degeneracy of the permutation τ_i.

Bousquet-Mélou–Schaeffer numbers enumerate decompositions of a permutation of a given cyclic type into the product of m arbitrary permutations.

- $C_\mu/|C_\mu|$ form an additive basis in the vector space $Z\mathbb{C}S_n$,
- characters χ_μ form an idempotent basis in the $Z\mathbb{C}S_n$,
- Schur functions s_μ form another basis in the $Z\mathbb{C}S_n$.

Definition

One-part Schur function s_k is the coefficient of t^k in the series $\exp\left(\sum_{i=0}^{\infty} \frac{p_i t^i}{i}\right)$.
The *Schur function* $s_\mu(p_1, p_2, \ldots)$ is the determinant of the matrix formed by one-part Schur functions:

$$s_\mu(p_1, p_2, \ldots) = \det(s_{\mu_i - i + j})_{1 \le i, j \le l(\mu)}$$

- BMS number $b_{m,k,\mu}$ is the coefficient of $C_\mu/|C_\mu|$ in the decomposition of $(\sum_{\nu\vdash n} t^{k(\nu)}C_\nu)^m/n!$ in the basis C_μ,
- generalized Hurwitz number $a_{m,k_1,\dots,k_m,\mu}$ is the coefficient of $C_\mu/|C_\mu|$ in the decomposition of $(C^{(k_1)}\dots C^{(k_m)})/n!$ in the basis C_μ.

- BMS number $b_{m,k,\mu}$ is the coefficient of $C_\mu/|C_\mu|$ in the decomposition of $(\sum_{\nu\vdash n} t^{k(\nu)} C_\nu)^m/n!$ in the basis C_μ,
- generalized Hurwitz number $a_{m,k_1,\ldots,k_m,\mu}$ is the coefficient of $C_\mu/|C_\mu|$ in the decomposition of $(C^{(k_1)} \ldots C^{(k_m)})/n!$ in the basis C_μ.

Proposition

- *Let* $a \in Z\mathbb{C}S_n$ *and* $a = \sum_{\mu\vdash n} a_\mu \frac{C_\mu}{|C_\mu|}$,
- *let* $f_a(\mu)$ *is an eigenvalue of* a *with respect to the basis* χ_μ,

then

$$\sum_{\mu\vdash n} a_\mu p_{\mu_1} p_{\mu_2} = \sum_{\mu\vdash n} f_a(\mu) \dim_\mu s_\mu(p)$$

$$A^{\circ}(u_1,\dots,u_m,p_1,p_2\dots) = \sum_{\mu,k_1,\dots,k_m} a^{\circ}_{k_1,\dots,k_m,\mu} u_1^{k_1}\dots u_m^{k_m} p_{\mu_1} p_{\mu_2}\dots =$$

$$= \langle \text{proposition 1} \rangle = \sum_{\mu} f_{C^{(k_1)}\dots C^{(k_m)}}(\mu) \frac{\dim_\mu}{|\mu|!} s_\mu(p).$$

Jucys-Murphy elements

Definition

$$X_k = (1,k) + (2,k) + \ldots + (k-1,k),\ k = 1,2,\ldots,n$$

Proposition (Jucys, Murphy, Okunkov-Vershik (1974, 1981, 2005))

1. The elements $X_1, \ldots, X_n$ commutes pairwise, and hence generate a commutative subalgebra in $\mathbb{C}S_n$.
2. Each element from the $Z\mathbb{C}S_n$ can be represented as a symmetric polynomial in the J-M elements.
3. If an element $a \in Z\mathbb{C}S_n$ is represented as a symmetric polynomial in the J–M elements, $a = P_a(X_1, \ldots, X_n)$, then the eigenvalue $f_a(\mu)$ of the action of a in a representation μ is equal to the value of the symmetric function P on the content $c(\mu)$ of the Young diagram μ.

$$C^{(k)} = \sigma_k(X_1, \ldots, X_n),$$

Just because each permutation admits a unique representation as a product of strictly monotonic transpositions: $(a_1, b_1)(a_2, b_2)\ldots(a_k, b_k),\ a_i < b_i, b_i < b_j$.

$$C^{(k)} = \sigma_k(X_1, \dots, X_n),$$

Just because each permutation admits a unique representation as a product of strictly monotonic transpositions: $(a_1, b_1)(a_2, b_2)\dots(a_k, b_k),\ a_i < b_i, b_i < b_j$.

$$n = 3,\ k = 2 \ \Rightarrow \ C^{(k)} = (1,2,3) + (1,3,2),$$

$$\sigma_2(X_1, X_2, X_3) = \sigma_2((1,2),(1,3)+(2,3)) = (1,2)(1,3)+(1,2)(2,3).$$

$$C^{(k)} = \sigma_k(X_1, \dots, X_n),$$

Just because each permutation admits a unique representation as a product of strictly monotonic transpositions: $(a_1, b_1)(a_2, b_2)\dots(a_k, b_k),\ a_i < b_i, b_i < b_j$.

$$n = 3,\ k = 2 \ \Rightarrow\ C^{(k)} = (1, 2, 3) + (1, 3, 2),$$

$$\sigma_2(X_1, X_2, X_3) = \sigma_2((1, 2), (1, 3)+(2, 3)) = (1, 2)(1, 3)+(1, 2)(2, 3).$$

$$\sum_{k=0}^{n} C^{(k)} u^k = \prod_{i=1}^{n} (1 + X_i u)$$

Generalized Hurwitz numbers

$$A^{\circ}(u_1,\dots,u_m,p_1,p_2\dots) = \sum_{\mu} f_{C^{(k_1)}\dots C^{(k_m)}}(\mu)\frac{\dim_\mu}{|\mu|!}s_\mu(p) =$$
$$= \sum_{\mu}\prod_{w\in\mu}\prod_{i=1}^{m}(1+c(w)u_i)\frac{\dim_\mu}{|\mu|!}s_\mu(p),$$

Generalized Hurwitz numbers

$$A^{\circ}(u_1,\dots,u_m,p_1,p_2\dots)=\sum_{\mu} f_{C^{(k_1)}\dots C^{(k_m)}}(\mu)\frac{\dim_{\mu}}{|\mu|!}s_{\mu}(p)=$$

$$=\sum_{\mu}\prod_{w\in\mu}\prod_{i=1}^{m}(1+c(w)u_i)\frac{\dim_{\mu}}{|\mu|!}s_{\mu}(p),$$

BMS numbers

$$B^{\circ}_{m}(u,p_1,p_2\dots)=\sum_{k=0}^{\infty}\sum_{\mu}u^k b^{\circ}_{m,k,\mu}p_{\mu_1}p_{\mu_2}\dots=$$

$$=\sum_{\mu}\prod_{w\in\mu}(1+c(w)u)^m\frac{\dim_{\mu}}{|\mu|!}s_{\mu}(p).$$

Definitions

- The space of *fermions* is the semi-infinite wedge product $\Lambda^{\frac{\infty}{2}} V$ — vector space freely spanned by the vectors $\underline{i_1} \wedge \underline{i_2} \dots, \; i_1 > i_2 > \dots, \; i_i \in \mathbb{Z} + \frac{1}{2}, \; i_k + k - \frac{1}{2} = c$,
- $(\cdot, \cdot)$ — scalar product with respect to this basis,
- $c = 0$ — zero-charge subspace $\mathcal{V}_0$,
- $\mathcal{V}_0$ freely spanned by the vectors of the form $v_\mu = \underline{\mu_1 - \frac{1}{2}}, \underline{\mu_2 - \frac{3}{2}}, \dots$, where $\mu = (\mu_1, \mu_2, \dots)$,
- $v_\emptyset = \underline{-\frac{1}{2}}, \underline{-\frac{3}{2}}, \dots$
- the *vacuum expectation value* of an operator P acting on $\mathcal{V}_0$: $\langle P \rangle = (v_\emptyset, P v_\emptyset)$.

Operators on the $\Lambda^{\frac{\infty}{2}} V$

- $\psi_k : \underline{i_1} \wedge \underline{i_2} \ldots \mapsto \underline{k} \wedge \underline{i_1} \wedge \underline{i_2} \ldots$
- $E_{i,j} = \begin{cases} \psi_i \psi_j^*, & \text{if } j > 0 \\ -\psi_j^* \psi_i, & \text{if } j < 0 \end{cases}$
- $\alpha_n = \sum_{i \in \mathbb{Z} + \frac{1}{2}} E_{i-n,i}$
- $\mathcal{F}_2 = \sum_{i \in \mathbb{Z} + \frac{1}{2}} \frac{i^2}{2} E_{i,i}$
- $\mathcal{E}_0(z) = \sum_{i \in \mathbb{Z} + \frac{1}{2}} e^{zi} E_{i,i}$
- function $\zeta(z) = e^{z/2} - e^{-z/2}$

Comment 1

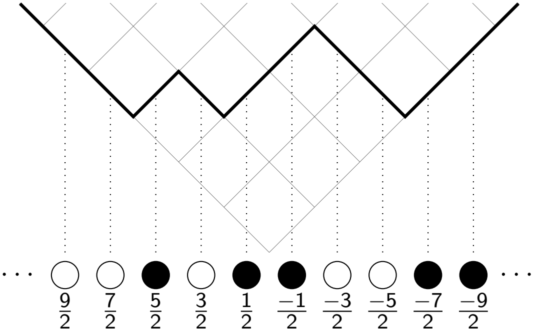
Operators $E_{i,j}$ preserve the charge. Describe the action on $\mathcal{V}_0$:

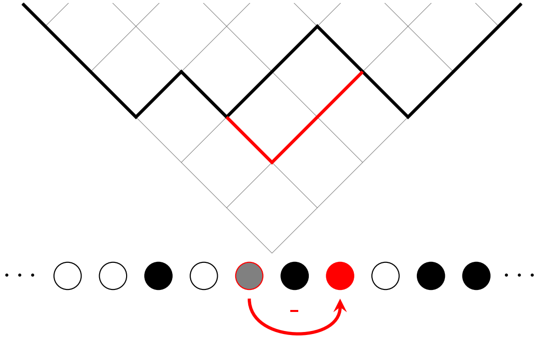
$$\begin{cases} i \neq j & \Rightarrow \text{if } v_\mu \text{ contents } \underline{j}, \text{ then replace it by } \underline{i}, \text{ otherwise } 0; \\ i = j > 0 & \Rightarrow \text{if } v_\mu \text{ contents } \underline{j}, \text{ then } E_{i,j}(v_\mu) = v_\mu, \text{ otherwise } 0; \\ i = j < 0 & \Rightarrow \text{if } v_\mu \text{ doesn't content } \underline{j}, \text{ then } E_{i,j}(v_\mu) = -v_\mu, \text{ otherwise} \end{cases}$$

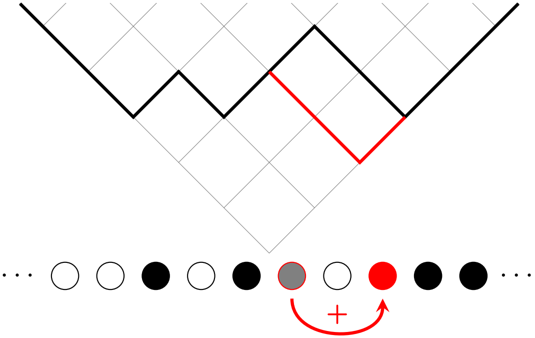
Comment 2

Action α_n on v_μ similar to the Murgnaghan-Nakayama rule.

$$\alpha_2\, v_{3,2,2} = v_{3,2} - v_{3,1,1}.$$


$\cdots$
$\frac{9}{2}$ $\frac{7}{2}$ $\frac{5}{2}$ $\frac{3}{2}$ $\frac{1}{2}$ $\frac{-1}{2}$ $\frac{-3}{2}$ $\frac{-5}{2}$ $\frac{-7}{2}$ $\frac{-9}{2}$
$\cdots$





Definition

The space of *bosons* is the space of power series in infinitely many variables $p_1, p_2, \ldots$

Bijection

Zero-charge subspace $\mathcal{V}_0$ and the space of bosons: $v_\mu \mapsto s_\mu$

Consequence of the Comment 2

$\prod_{i=1}^{l(\mu)} \alpha_{-\mu_i} |0\rangle$ formed another basis in the space $\mathcal{V}_0$

Operator formula for BMS numbers

Proposition (Kramer-Lewanski-Shadrin, 2016)

$$D^{\sigma}(u) = \exp\left(-\left(\frac{\mathcal{E}_0\left(-u^2\frac{d}{du}\right)}{\zeta\left(-u^2\frac{d}{du}\right)} - \sum_{i\in\mathbb{Z}+\frac{1}{2}} iE_{i,i}\right)\log u\right)$$

Consequence

$$B_m^{\circ}(u) = \left\langle e^{\alpha_1}\left(D^{\sigma}(u)\right)^m \prod_{i=1}^{l(\mu)} \frac{\alpha_{-\mu_i}}{\mu_i}\right\rangle$$

BMS numbers

Genus 0

$$b_{m,2n-2+l(\mu),\mu} = \prod_{i=1}^{l(\mu)} \frac{(m\mu_i - 1)_{\mu_i}}{\mu_i!} \cdot m((m-1)n-1)_{l(\mu)-3}$$

Genus 1

$$b_{m,2-n+l(\mu),\mu} = \prod_{i=1}^{l(\mu)} \frac{(m\mu_i - 2)_{\mu_i - 1}}{\mu_i!} \cdot \frac{m}{24} P_{2n-1}(\mu, m)$$

Thank you!