International Conference of Mathematical Physics “Kezenoi-Am 2017”

Faculty of Mathematics and Computer Technologies
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Welcome Address

Welcome to the International Conference of Mathematical Physics “Kezenoi-Am 2017”! This a preliminary version of the conference abstracts book.

Abstracts

Zeros of polynomials and solvable nonlinear evolution equations

Francesco Calogero

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There exist convenient explicit formulas expressing the time-derivative (of any order k) of each of the N zeros of a (monic) time-dependent polynomial of degree N in terms of: (i) the time-derivatives of the same order k of the N coefficients of that polynomial; and (ii) the N zeros of that polynomial and their time-derivatives of order less than k. This allows the identification of many *new* dynamical systems—*solvable/integrable* by *algebraic* operations—including N-body problems characterized by Newtonian ("accelerations equal forces") equations of motion describing N, nonlinearly interacting, point-particles moving in the *complex* plane, or equivalently in the *real* Cartesian plane. The same approach can also be used to identify new solvable systems of nonlinear PDEs; as well as a new differential algorithm to compute all the zeros of generic polynomials of arbitrary degree N. This approach moreover yields infinite hierarchies of all these solvable nonlinear evolution equations, via the introduction of the notion—of mathematical interest in its own right—of generations of monic polynomials, all of the same arbitrary degree N, with the N coefficients of the polynomials of the next generation coinciding with every one of the $N!$ permutations of the N zeros of each polynomial of the current generation. In this talk I will describe these recent developments, which have been reported in several papers published in the last 1-2 years—some authored by myself alone and some co-authored with **Oksana Bihun**, with **Mario Bruschi** or with **François Leyvraz**—and which are reviewed in a 170-page book I just completed [18] (see Bibliography below).

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Decorated character varieties and monodromy data surfaces associated with Painlevé confluented equations

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We discuss a definition of decorated character varieties, related to representations of fundamental grouppoids constructed for affine cubic surfaces which are playing the role of monodromy data and are compatible with the pole coalescence procedure. We construct Poisson algebra structures on these surfaces and discuss various "quantisation" procedures for the algebras.
Acknowledgements: My talk is based on joint works (partly, in progress) with Marta Mazzocco (arXiv:1511.03851) and Leonid Chekhov (arXiv:1705.01447).

Cayley-Hamilton Identity and Drinfeld-Sokolov reduction in quantum algebras

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I plan to exhibit different forms of the CH identity in some quantum algebras. In particular, I'll consider the so-called braided Yangians recently introduced by myself (jointly with Saponov). A quantum counterpart of the Drinfeld-Sokolov reduction based on the CH identity will be discussed as well.

The three-dimensional structure in magnets

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The report discusses the analytical and numerical methods for calculating new structures in magnetic materials. Spatial structures in the single-exchange approximation multisublattice antiferromagnetic described Lagrangian density:

$$L = \frac{1}{2}\,\mathrm{Sp}\Big[\frac{\partial G^{-1}}{\partial x_\mu}\frac{\partial G}{\partial x^\mu}\Big], \qquad (1)$$

which depends on the coordinates of the matrix G is an element of group SU(2). We use three substitutions which reduced system of equations for(1) to the new system with a simple geometrical interpretation, and we use differential-geometric integration method . The first of these systems is trigonal system for harmonic conversion coordinate. Now derivatives new fields are defined as variables related to the tensor components of the metric induced by such transformation. Then the required equation can be rewritten in terms of the metric tensor components. Since the independent variables were first Euclidean, then the curvature tensor introduced metric is zero. As a result, we get a self-consistent system of equations for determining the components of the metric tensor. When this equation is zero curvature main equation, and the required system of equations it reductions. Their solution allows further the formulas of classical geometry to find a solution to the desired differential equation in the form of implicit functions. Such a differential-geometric approach offers a broad class of three-dimensional structures, the preparation of which other methods are extremely difficult. As a result we find many spatial textures comprising vortices solitons spatial sources antiferromagnetic “target”, spiral vortices and their dipole configuration antiferromagnetic “target”, spiral vortices and their dipole configuration, and the structure with a degree of mapping is equal to unity, similar in some of its properties with topological solitons.

Developed new methods and numerical algorithms based on massive parallel computing technology NVIDIA CUDA search of stable and metastable states allowed to find a new type of thermodynamically stable magnetic states in chiral magnets. Such structures are solitons micromagnetic equations localized in three dimensions of the sample near the free boundary. The stability of the new particle-state due to substantial energy barrier created by

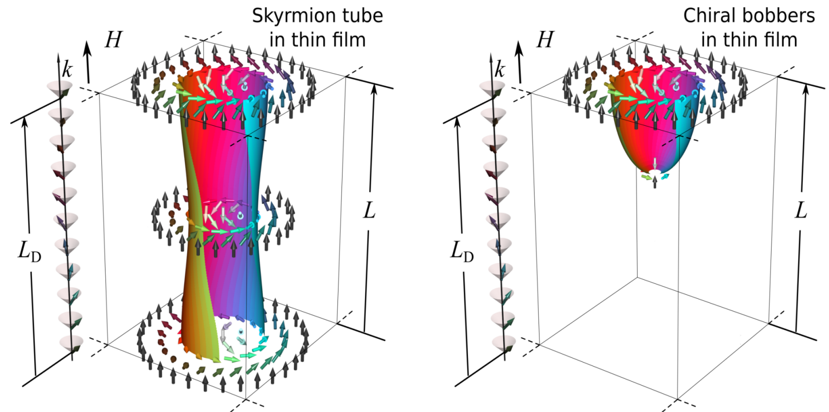

Figure 1: The structure of the magnetization vector of the vortex tube, i.e. chiral skyrmion, and the new stable state in the film of the helical magnet.

chiral twisting surface magnetization vector. This barrier keeps Bloch point near the free surface of the sample at a certain optimal depth P (Fig. 1). Because of the leading role of chirality for the emergence of such a state and location close to the surface, like the fishing float on the water surface, for this object, we have introduced the notation chiral float.

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3-d integrable statistical models and invariants in low dimensional topology

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The talk is focused on some particular family of statistical models on 6-valent graphs and their relations with the theory of invariants of 2-knots, i.e. the isotopy classes of S^2 embeddings in $\mathbb{R}^4$, and integrable 3-d models of statistical physics on regular latices. In both cases the integrability properties are provided by the higher homotopic algebraic structure related with the Zamolodchikov tetrahedron equation.

I will briefly recall some facts related with the problem of constructing invariants of 2-knots. This issue is principally due to the work of Carter, J.S., Jelsovsky, D., Langford, L., Kamada, S., Saito, M., and is based on quandle cohomology. In our works (with G.I. Sharygin and I.G. Korepanov) we extend in some sense their approach using the algebraic structure underlying the tetrahedral equation. The role of the tetrahedral cohomology will be especially emphasized.

Another interesting aspect of this construction is the integrability of the considered statistical model when restricted to the regular periodic 3-dimensional lattice. I will comment on the relation of the subject with the famous Maillet result in 2-d lattices.

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The Integrable Case of Adler – van Moerbeke. Spectral Curve, Discriminant Set and Bifurcation Diagram

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In 1986 Adler and van Moerbeke discovered the general case of integrability on the Lie algebra $so(4)$. An explicit form of the additional integral was presented in the original paper [1]. Later Reyman and Semenov-Tian-Shansky [2], with the help of a special algebra $\mathfrak{g}_2$, gave the Lax representation $\dot{L}(\lambda) = [L(\lambda), A(\lambda)]$. Other additional integrals different from that in [1] were presented by Bolsinov and Borisov [3] and V. Sokolov [4].

From the mechanical point of view the case of Adler – van Moerbeke can be reduced to the system governed by the Euler equations

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}}, \quad \dot{\boldsymbol{S}} = \boldsymbol{S} \times \frac{\partial H}{\partial \boldsymbol{S}} \tag{1}$$

which describe the motion of a rigid body with an ellipsoidal cavity filled by a perfect incompressible vortical fluid around a fixed point. Here the 3-dimensional vector $\boldsymbol{M}$ denotes the angular momentum of the 'body+fluid' system and the components of $\boldsymbol{S}$ are proportional to the fluid's vorticity.

The Hamiltonian H is the kinetic energy of the 'body+fluid' system expressed in terms of $(\boldsymbol{M}, \boldsymbol{S})$

$$H = (\boldsymbol{M}, A\boldsymbol{M}) + 2(\boldsymbol{M}, B\boldsymbol{S}) + (\boldsymbol{S}, C\boldsymbol{S}).$$

Here A, B and C are diagonal 3×3 matrices which read

$$\begin{aligned}
&A = \operatorname{diag}[\alpha_2^2\alpha_3^2, \alpha_1^2\alpha_3^2, \alpha_1^2\alpha_2^2]\,;\\
&B = \operatorname{diag}[(\alpha_1-\alpha_2)(\alpha_3-\alpha_1)\alpha_2\alpha_3, (\alpha_2-\alpha_1)(\alpha_3-\alpha_2)\alpha_1\alpha_3, (\alpha_3-\alpha_1)(\alpha_2-\alpha_3)\alpha_1\alpha_2]\,;\\
&C = \operatorname{diag}[\alpha_2\alpha_3(\alpha_2\alpha_3-4\alpha_1^2), \alpha_1\alpha_3(\alpha_1\alpha_3-4\alpha_2^2), \alpha_1\alpha_2(\alpha_1\alpha_2-4\alpha_3^2)]\,.
\end{aligned}$$

Besides the energy integral H, the equations (1) always have the geometric integrals

$$F_1 = (\boldsymbol{M}, \boldsymbol{M}), \qquad F_2 = (\boldsymbol{S}, \boldsymbol{S}),$$

which are the Casimir functions with respect to the Lie-Poisson bracket

$$\{M_i, M_j\} = \varepsilon_{ijk}M_k, \quad \{M_i, S_j\} = 0, \quad \{S_i, S_j\} = \frac{1}{3}\varepsilon_{ijk}S_k.$$

On the common level

$$\mathcal{P}_{a,b} = \{F_1 = a^2,\ F_2 = b^2\} \cong S^2 \times S^2$$

the induced Lie-Poisson bracket is non-degenerate and the system (1) restricted to this level gives an integrable Hamiltonian system with two degrees of freedom and with an additional integral K of the form

$$\begin{aligned}
K = 3\sum_{i,j} \alpha_i(\alpha_j-\alpha_i)M_jS_jS_i^2 + \sum_i (\alpha_i-\alpha_j)(\alpha_i-\alpha_k)M_iS_i^3 -\\
-(\boldsymbol{M}, \boldsymbol{M})\sum_i [\alpha_j\alpha_kM_iS_i + 2(\alpha_j^2+\alpha_k^2)S_i^2].
\end{aligned}$$

If $\alpha_1 + \alpha_2 + \alpha_3 = 0$, then the integral K is in involution with Hamiltonian H.

It is well known that the invariants of the matrix $\mathrm{Tr}L(\lambda)^k$ are first integrals. This integrals generate a momentum map $\mathcal{F}$. At present we do not have a general theorem that links the structure of the bifurcation diagram (the image of the critical points of the momentum map) to the discriminant set of the algebraic curve $\Gamma(\lambda, \mu) = \det(L(\lambda) - \mu I)$. However as we can see from the study of specific mechanical systems [5], [6] such a link exists and it can be used as a hypothesis for the derivation of the equations of bifurcation diagram (with a subsequent proof of sufficiency).

Here, for the Adler – van Moerbeke case, we explicitly present the spectral curve $\Gamma(\lambda, \mu)$. This enables us upon the inspection of the curves singularities to find the bifurcation diagram of the momentum map $\mathcal{F}$. Here we also discuss the phase topology of that Hamiltonian system. In particular we find the bifurcation diagram of the momentum map and explore bifurcations of the Liouville tori. Examples of the bifurcation diagram singularities of rank 0 and some other details is presented in [7].
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Vector Integrable Equations

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Our talk is devoted to differential equations with two independent variables in a vector space of arbitrary dimension. We consider both evolution and hyperbolic equations that have infinitely many higher conservation laws. A component-less version of the symmetry approach to integrability is described, some examples of integrable equations are presented and several classification results are formulated.

Public-key cryptography and integrable systems.

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Algebraic geometry is one of the central subjects of mathematics and intersection theory is at the heart of algebraic geometry. It has many applications in classical mechanics, mathematical physics, complex analysis, homotopy theory, symplectic geometry, representation theory, Gromov-Witten theory, theory of virtual fundamental cycles, quantum intersection rings. In a different direction, there are many practical applications of the intersection theory in computer science, for instance in modern computer cryptography, computer graphics, computer vision, in mathematical and analytical software etc.

We use a language of intersection theory in order to apply some well-known algorithms of computer science to construct new integrable bi-Hamiltonian systems. For instance, in 1987 Cantor was the first to give a concrete algorithm for performing computations in Picard groups of hyperelliptic curves. In hyperelliptic curve cryptography this algorithm yields coding/decoding operations, whereas in classical mechanics the same algorithm yields integrable discrete maps, Bäcklund transformations or integrable discretizations of continuous dynamical systems integrable by Abel quadratures on the underlying hyperelliptic curve.

When we implement a hyperelliptic curve cryptosystem in a real security system, we have to prepared for various cryptographic attacks. Some of these attacks are based on the symmetry of intersection of two irreducible smooth projective curves. We can also apply the corresponding algorithms to construct auto and hetero Bäcklund transformaions. As an example we show how to construct new bi-Hamiltonian systems with natural Hamilton function and integrals of motion of third, fourth and sixth order in momenta using some standard cryptographic protocols.

An asymptotic method for homogenization for a generalized system of Beltrami operators

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Let Q be a simply connected bounded domain with smooth boundary ($\partial Q \in C^2$). Consider the following Riemann-Hilbert problem:

$$\begin{cases} A_\varepsilon u_\varepsilon \equiv \partial_{\bar z} u_\varepsilon + \mu^\varepsilon \, \partial_z u_\varepsilon + \nu^\varepsilon \, \partial_{\bar z} \overline{u}_\varepsilon = f \in L_2(Q), \\ u_\varepsilon \in W_0(Q) \equiv \left\{ u \in W_2^1(Q) \mid \operatorname{Re} u\big|_{\partial Q} = 0, \ \int_Q \operatorname{Im} u \, dx = 0 \right\}, \end{cases} \tag{2}$$

where $0 < \varepsilon < 1$, $\mu^\varepsilon = \mu(\varepsilon^{-1}x)$ and $\nu^\varepsilon = \nu(\varepsilon^{-1}x)$; $\mu(x) = \mu(x_1, x_2)$, $\nu(x) = \nu(x_1, x_2)$; here $\mu(x) = \mu(x_1, x_2)$ and $\nu(x) = \nu(x_1, x_2)$ are bounded measurable periodic complex-valued functions (with period T in each variable) satisfying the ellipticity condition:

$$\operatorname*{vrai\,sup}_{x \in \mathbb{R}^2} \left(|\mu(x)| + |\nu(x)| \right) \leqslant k_0 < 1,$$

and $k_0 > 0$ is a positive constant. We treat our spaces as spaces over $\mathbb{R}$.

The expression $\|u\|_{W_0(Q)} = \|\partial_{\bar z} u\|_{L_2(Q)}$, $u \in W_0(Q)$, defines a norm in $W_0(Q)$, which is equivalent to the original norm in $W_2^1(Q)$.

It is known that the Riemann-Hilbert problem (2) has a unique solution for each right-hand side $f \in L^2(Q)$, and $u_\varepsilon \to u^0$ in $L_2(Q)$ as $\varepsilon \to 0$, where u^0 solves the homogenized problem

$$A_0 u^0 \equiv \partial_{\bar z} u^0 + \mu^0 \, \partial_z u_\varepsilon + \nu^0 \, \partial_{\bar z} \overline{u^0} = f \in L_2(Q), \quad u^0 \in W_0(Q).$$

The coefficients μ^0, ν^0 of the homogenized operator are the constants defined by $\mu^0 = \langle \mu\,\mathcal{Q} + \overline{\nu}\,\mathcal{P}\rangle$, $\nu^0 = \langle \overline{\mu}\,\mathcal{P} + \nu\,\mathcal{Q}\rangle$, where $\mathcal{P} = 2^{-1}(p_1 + ip_2)$, $\mathcal{Q} = 2^{-1}(\overline{p_1} + i\,\overline{p_2})$ and p_1, p_2 belong to the kernel of the operator A^* adjoint to the operator of the periodic boundary value problem: $\partial_{\overline{z}}u + \mu\,\partial_z u + \nu\,\partial_{\overline{z}}\overline{u} = f \in L_{2,\mathrm{per}}$, $u \in W^1_{2,\mathrm{per}}$; $\langle g\rangle$ is the mean value of the periodic function g.

The following result holds.

Theorem. *Assume that the right-hand side f of Riemann-Hilbert problem (2) belongs to $W^1_2(Q)$ and let Q be a simply connected domain with (C^2-)smooth boundary. Then*

$$\|u_\varepsilon - u_1^\varepsilon\|_{W_2^1(Q)} \leqslant c\sqrt{\varepsilon}\,\|f\|_{W_2^1(Q)}, \qquad \|u_\varepsilon - u^0\|_{L_2(Q)} \leqslant c\sqrt{\varepsilon}\,\|f\|_{W_2^1(Q)}, \quad (3)$$

where c is a positive constant independent of ε and f; $u_1^\varepsilon(x) = u^0(x) + \varepsilon(N(y)\partial_z u^0(x) + M(y)\partial_{\overline{z}}\overline{u^0(x)})$, $y = \varepsilon^{-1}x$; $N(y) = N_1(y) - iN_2(y)$, $M(y) = N_1(y) + iN_2(y)$, and $N_1(y)$, $N_2(y)$ are the periodic solutions of the problems $\partial_{\overline{z}}N_j + \mu\,\partial_z N_j + \nu\,\partial_{\overline{z}}\overline{N}_j = \chi_j$, $j = 1, 2$, where $\chi_1 = \frac{1}{2}\,(\mu^0 + \nu^0 - \mu(x) - \nu(x))$, $\chi_2 = \frac{i}{2}\,(\mu^0 - \nu^0 - \mu(x) + \nu(x))$.

The first inequality in (3) gives an estimate of the difference between the exact solution of (2) and the first approximation; the second gives an estimate of the rate of convergence of the exact solution to the solution of the homogenized problem.

Similar result holds for the generalized system of Beltrami operators.

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Naturally graded Lie algebras of slow growth

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The growth of a finitely generated infinite-dimensional Lie algebra $\mathfrak{g}$ can be described by the Gelfand-Kirillov dimension

$$GK \dim \mathfrak{g} = \limsup_{n\to\infty} \frac{\log \dim V^n}{\log n},$$

where V^n is the subspace in $\mathfrak{g}$ spanned by all elements of length at most n with arbitrary arrangements of brackets. Shalev and Zelmanov studied [2] Lie algebras of GK-dimension 1. In particulat they considered a special subclass of graded two-generated Lie algebras with the slowest possible growth ($\dim V^n = n+1$), the so-called Lie algebras of maximal class.

A Lie algebra $\mathfrak{g}$ is called naturally graded if it is isomorphic to $\mathrm{gr}_C\mathfrak{g}$, its associated graded Lie algebra with respect to the filtration by ideals $C^i\mathfrak{g}$ of the descending central sequence.

We classify [1] naturally graded Lie algebras $\mathfrak{g} = \oplus_{i\in\mathbb{N}}\mathfrak{g}_i$ with the following linear growth:

$$\dim V^n \leq \frac{3}{2}n+\frac{1}{2}.$$

The relations with characteristic Lie rings [3] of the Klein-Gordon equation $u_{xy} = f(u)$ will be discussed.

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The combinatoric formulas for the characteristic classes of triangulated bundles

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The notion of higher torsion invariants is an analog of the well-known Franz-Reidemeister torsion, introduced in 1930-ies, which allows distinguish homotopy-equivalent, but non-homeomorphic manifolds. Later (in 1970-ies) it was shown by Cheeger and Müller that this invariant can be described analytically, namely that it coincides with the Ray-Singer analytic torsion.

Suppose now that we are given a smooth manifold fibre bundle $E \to B$ (with manifold fibre F). In this case Igusa, Klein and others introduced the notion of higher Franz-Reidemeister torsion, which is now a series of real cohomology classes of B. Loosely speaking, it is the fibrewise Reidemeister torsion of E, taken in derived sense (there also exist analytic and homotopy theoretic variants of this torsion, developed by Bismut, Lott, Dwyer, Weiss, Williams and others). One of the important properties of this torsion is that when applied to the spherical bundles of complex vector bundles it gives (up to a coefficient) Chern classes of the original bundle.

Motivated by this property, we are going to apply these constructions to the following problem: suppose we have a simplicial map between a pair of simplicial complexes $p : K \to L$, such that the geometric realization of it is a locally trivial spheric bundle of a complex or real oriented vector bundle. Then the question is to describe the characteristic classes of the original vector bundle from the combinatoric structure of the map p. It is this question, that I shall address in my talk, based on the joint work (in progress) with Nikolai Mnëv (POMI).

If the dimension of the fibre of p is equal to 1, there is an explicit combinatoric way to compute the coefficients of the cocycles, representing the first Chern class of the bundle:

$$c_1(\sigma) = \frac{1}{2N_\sigma}(\#(\text{positive triangles} - \#(\text{negative triangles})),$$

where for any 2-simplex $\sigma \in L$, N_σ is the number of vertices in the bundle K over σ, and positive and negative triangles are the triangles spanned by the vertices in this restriction, which fall under the action of p onto the vertices of σ, so that their order correspond to the orientation of the fibre, or is opposite to it.

However, in case of higher dimensions, we still don't know a similarly efficient recipe, though we hope that one can advance with the help of torsion invariants. In my talk I shall briefly describe the basic constructions of the higher torsion (more accurately, of its analytic version due to Bismut and Lott). Then I will describe a combinatoric algorithm, which allows one transfer the main steps of that construction to the case we consider. We suppose, that the resulting cohomology classes are also proportional to Chern (or Pontrjagin) classes of the original bundle.

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POSITIVITY CRITERIONS OF CONVOLUTION INTEGRAL AND INTEGRO-DIFFERENTIAL OPERATORS AND THEIR APPLICATIONS.

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Various classes of nonlinear integral equations of convolution type on a finite (in the periodic case) and an infinite interval of integration were studied in the monograph [1]. There an essential role was played by the positivity property (in the sense Bochner) of convolution-type operators, a property guaranteed by the nonnegativity condition for the discrete (in the periodic case where the interval of integration is the closed interval $[-\pi, \pi]$) or the integral (in the case where the interval of integration is the whole real axis or semiaxis) Fourier cosine transform of its kernel. In this work we establish that the convolution integro-differential operator is positive if and only if the Fourier sine transform of the kernel is a nonnegative function on the half-line. By using this result, global theorem on the existence and uniqueness of solution for various classes of nonlinear integro-differential convolution-type equations in the real Lebesgue spaces $L_p(\mathbb{R})$ and $L_p(-\pi, \pi)$ are proved using the method of maximal monotone operators [2].

Hereafter we assume that given function $F(x, u)$ generating nonlinearity in the considered equation is defined for $x, u \in \mathbb{R}$ and satisfies Caratheodory conditions: it is measurable in x for each fixed u and is continuous in u for almost each x. We denote by $L_p^+(\mathbb{R})$ the set of all non-negative functions in $L_p(\mathbb{R})$, while F stands for the superposition operator (Nemytskii operator) generated by function $F(x, u)$.

In particular, we have the following theorem.

Theorem. *Let* $1 < p < \infty$, $f(x) \in L_{p'}(\mathbb{R})$, $p' = p/(p-1)$, *kernel* $h(x) \in L_1(\mathbb{R})$ *and*

$$\int_{-\infty}^{\infty} h(t) \cdot \sin(xt)dt \geq 0, \quad \forall x \in [0, \infty)\,.$$

If for almost all $x \in \mathbb{R}$ *and all* $u \in \mathbb{R}$ *the nonlinearity* $F(x, u)$ *satisfies the conditions*

1). $|F(x,u)| \le a(x) + d_1|u|^{p-1}$, where $a(x) \in L^+_{p'}(\mathbb{R})$, $d_1 > 0$;
2). $F(x,u)$ is a nondecreasing function of u;
3). $F(x,u)\cdot u \ge d_2|u|^p - D(x)$, where $D(x) \in L^+_1(\mathbb{R})$, $d_2 > 0$,
then nonlinear integro-differential equation

$$F(x,u(x)) + \int\limits_{-\infty}^{\infty} h(x-t)u'(t)dt = f(x)$$

has a solution $u(x) \in L_p(\mathbb{R})$ with $u'(x) \in L_{p'}(\mathbb{R})$. This solution is unique if $F(x,u)$ strictly increases in u for almost each fixed $x \in \mathbb{R}$.

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Quazipolynomiality of Bousquet-Melou–Schaeffer numbers.

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Simple Hurwitz number $h_{m,\mu}$ defined by the following equation

$$h^{\circ}_{m,\mu} = \frac{1}{|\mu|!}|\{(\tau_1, \ldots, \tau_m), \tau_i \in C_2(S_{|\mu|})|\tau_m \circ \ldots \circ \tau_1 \in C_\mu(S_{|\mu|})\}|.$$

Here μ is the partition of the $|\mu|$, $S_{|\mu|}$ is the permutation group with $|\mu|$ elements, $C_2(S_{|\mu|})$ is the set of all transpositions in $S_{|\mu|}$ and $C_\mu(S_{|\mu|})$ is the set of all permutations with the cyclic type μ in the $S_{|\mu|}$.

Using the infinite wedge formalism one can rewrite Hurwitz numbers in terms of the vacuum expectation value of certain operators on the semi-infinte wedge space $\mathcal{V}$. This approach allows one to prove quasi-polynomiality property of simple Hurwtiz numbers independently from the ELSV formula and moreover allows one to give a new proof of the ELSV formula [1].

Bousquet-Mélou–Schaeffer numbers $b_{m,k;\kappa}$ are defined by the following identity:

$$b_{m,k;\mu} = \frac{1}{|\mu|!}|\{(\tau_1, \ldots \tau_m), \tau_i \in S_{|\mu|}| \sum_{i=1}^{m} k(\tau_i) = k, \tau_m \circ \ldots \circ \tau_1 \in C_\mu(S_{|\mu|})\}|,$$

where $k(\tau_i) = |\tau_i| - l(\tau_i)$ is the degeneracy of the permutation τ_i. In other words, Bousquet-Mélou–Schaeffer numbers enumerate decompositions of a permutation of a given cyclic type into the product of m arbitrary permutations.

The formalism based on Jucys-Murphy elements allows one to write down the generating function for Bousquet-Mélou–Schaeffer numbers and operators in the semi-infinite wedge space associated to Bousquet-Mélou–Schaeffer numbers. Therefore, we may predict quasi-polynomiality and existence of ELSV-type formulae for Bousquet-Mélou–Schaeffer numbers.

In the talk I will describe all objects, which are defined and not defined above and explain all steps of proving of quasi-polynomiality results. This is the work in progress with P. Dunin-Barkowski.

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Lyapunov spectrum of Markov and Euclid trees

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Markov triples are the integer solutions of the Markov equation

$$x^2 + y^2 + z^2 = 3xyz.$$

They surprisingly appeared in many areas of mathematics: initially in classical number theory, but more recently in hyperbolic and algebraic geometry, the theory of Teichmüller spaces, Frobenius manifolds and Painlevè equations.

We study the Lyapunov exponents $\Lambda(x)$ for Markov dynamics as a function of path determined by $x \in \mathbb{R}P^1$ on a binary planar tree, describing the growth of Markov triples and their "tropical" version - Euclid triples. We show that the corresponding Lyapunov spectrum is $[0, \ln\varphi]$, where φ is the golden ratio, and prove that on the set X of the most irrational numbers the corresponding function is convex and strictly monotonic. The key step is using the relation of Markov numbers with hyperbolic structures on punctured torus, going back to D. Gorshkov and H. Cohn, and, more precisely, the recent result by V. Fock in the theory of Teichmüller spaces [1].

Similar results are proved for the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz + D,$$

studied by Mordell and related to one-hole hyperbolic tori.

Acknowledgements: The talk is based on a joint work with K. Spalding[2].

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