Abel differential equations and Bäcklund transformations

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Jacobi method

The stationary Hamilton-Jacobi equation

$$H = E$$

is said to be separable in a set of canonical coordinates

$$\{u_i, p_{u_j}\} = \delta_{ij}\,, \qquad \{u_i, u_j\} = \{p_{u_i}, p_{u_j}\} = 0\,,$$

if there is an additively separated complete integral

$$W(u_1, \ldots, u_n; \alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} W_i(u_i; \alpha_1, \ldots, \alpha_n)$$

depending non-trivially on constants $\alpha_1, \ldots, \alpha_n$.

C.G.J. Jacobi, Vorlesungen über dynamik, Berlin, G.Reimer, 1884

"The main difficulty in integrating a given differential equation lies in introducing convenient variables, which there is no rule for finding. Therefore, **we must travel the reverse path** and after finding some notable substitution, look for problems to which it can be successfully applied."

Substituting canonical coordinates u, p_u into the separated relations

$$\Phi_i(u_i, p_{u_i}, H_1, \ldots, H_n) = 0 \qquad \text{with} \quad \det\left[\frac{\partial \Phi_i}{\partial H_j}\right] \neq 0$$

and solving the resulting equations with respect to $H_1, \ldots, H_n$, we obtain new integrable system with independent integrals of motion $H_1, \ldots, H_n$ in the involution.

Possible generalisation of Jacobi method

Let us add one more step to this well-known construction of new integrable systems:

- take some known Hamilton-Jacobi equation $H = E$ separable in canonical variables (u, p_u);
- make Bäcklund transformation (BT) of variables $(u, p_u) \to (\tilde{u}, \tilde{p}_u)$;
- substitute new canonical variables $\tilde{u}, \tilde{p}_u$ into the suitable separated relations and obtain new interesting integrable systems.

Kodama, Wadati (1976) and Wojciechowski (1982) proposed

Definition:

BT is canonical transformation of variables preserving the Hamiltonian character of the equations of motion and the form of Hamilton-Jacobi equations.

Integration of equations of motion is reduced to the calculation of integrals of algebraic functions

$$I = \int R(x, y)dx$$

- x - independent variable
- R - rational function on x and y
- y - algebraic function on x

Function $y = y(x)$ is defined by an irreducible equation of the form

$$f(x, y) = y^n + A_1 y^{n-1} + \cdots + A_n = 0.$$

where the $A_i(x)$ are polynomials in x

In modern terms, one considers the plane algebraic curve

$$\mathcal{C}: \qquad f(x,y)=0$$

rational differential

$$\omega = R(x,y)dx|_{\mathcal{C}}$$

and integral

$$I = \int_{(x_0,y_0)}^{(x,y)} \omega$$

on a compact Riemann surface.

Such integrals are highly transcendental functions of the upper limit of integration and consequently are generally difficult to study directly.

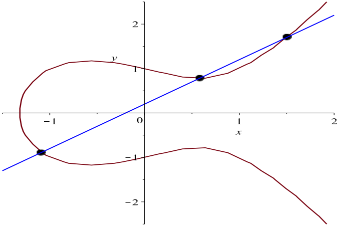
Abel's idea was to consider the sum of integrals

$$I(t) = \sum_{k=1}^{n} \int^{\mathrm{p}_k(t)} \omega\,,$$

to the variable points $\mathrm{p}_k(t) = (x_k(t), y_k(t))$ of intersection of $\mathcal{C}$ with a a family of plane curves, which are defined by equation

$$h(x,y,t) = 0$$

depending rationally on a parameter t.



Abel Theorem

If ω is a regular differential, which has no poles on C, then

$$I(t) = \sum_{k=1}^{n} \int^{\mathrm{p}_k(t)} = conts \quad \Rightarrow \quad I'(t) = 0\,.$$

or

$$\frac{d}{dt}\sum_{k=1}^{n} \int^{\mathrm{p}_k(t)} \omega = \omega(\mathrm{p}_1) + \omega(\mathrm{p}_2) + \cdots + \omega(\mathrm{p}_n) = 0\,.$$

where

$$\omega(\mathrm{p}_k) = R\big(x_k(t), y_k(t)\big)\, x_k'(t)\, dt$$

Equation of this form is called an Abel differential equation.

For instance, on the elliptic curve $\mathcal{C}$ we have three possible Abel equations

$$\frac{dx_1}{y_1}+\frac{dx_2}{y_2}=0\,,\qquad \frac{dx_1}{y_1}+\frac{dx_2}{y_2}+\frac{dx_3}{y_3}=0\,,$$

$$\frac{dx_1}{y_1}+\frac{dx_2}{y_2}+\frac{dx_3}{y_3}+\frac{dx_4}{y_4}=0$$

on four points of intersection of $\mathcal{C}$ with parabola, see details in
A.G. Greenhill, *The applications of elliptic functions*, 1892.

Modern theory of Abel differential equations may be found in

M. Green, P. Griffiths, *Abel's differential equations*, 2002;
P. Griffiths, *The Legacy of Abel in Algebraic Geometry*, 2004;
S. L. Kleiman, *The Picard scheme*, 2005.

Abel equations in Jacobi paper

Über eine neue Methode zur Integration der hyperelliptischen Differentialgleichungen und über die rationale Form ihrer vollständigen algebraischen Integralgleichungen.

(Von Herrn Professor Dr. *C. G. J. Jacobi* zu Berlin.)

Ich werde das System Differentialgleichungen

$$
1. \quad \begin{cases}
\dfrac{dx_1}{\sqrt{X_1}} + \dfrac{dx_2}{\sqrt{X_2}} \dots + \dfrac{dx_n}{\sqrt{X_n}} = 0, \\[2mm]
\dfrac{x_1\,dx_1}{\sqrt{X_1}} + \dfrac{x_2\,dx_2}{\sqrt{X_2}} \dots + \dfrac{x_n\,dx_n}{\sqrt{X_n}} = 0, \\[2mm]
\dfrac{x_1^2\,dx_1}{\sqrt{X_1}} + \dfrac{x_2^2\,dx_2}{\sqrt{X_2}} \dots + \dfrac{x_n^2\,dx_n}{\sqrt{X_n}} = 0, \\[2mm]
\dots\dots\dots\dots\dots \\[2mm]
\dfrac{x_1^{n-2}\,dx_1}{\sqrt{X_1}} + \dfrac{x_2^{n-2}\,dx_2}{\sqrt{X_2}} \dots + \dfrac{x_n^{n-2}\,dx_n}{\sqrt{X_n}} = 0,
\end{cases}
$$

in welchem $X_1, X_2, \dots X_n$ dieselben ganzen Functionen $2n^{\text{ten}}$ Grades respective von den Variabeln $x_1, x_2, \dots x_n$ sind, und $n > 2$ ist, mit dem Namen *eines Systems hyperelliptischer Differentialgleichungen* bezeichnen.

Following to Jacobi consider two Hamiltonian flows

$$\frac{du_1}{dt_1} = \{u_1, H_1\} = \frac{p_{u_1}}{u_1 - u_2}, \qquad \frac{du_2}{dt_1} = \{u_2, H_1\} = -\frac{p_{u_2}}{u_1 - u_2}$$

$$\frac{du_2}{dt_2} = \{u_1, H_2\} = -\frac{u_2 p_{u_1}}{u_1 - u_2}, \qquad \frac{du_2}{dt_2} = \{u_2, H_2\} = \frac{u_1 p_{u_2}}{u_1 - u_2}.$$

where

$$p_i^2 = f(u_i), \qquad f(u) = a_6 u^6 + a_5 u^5 + a_4 u^4 + a_3 u^3 + a_2 u^2 + a_1 u + a_0.$$

Standard combinations of these equations read as

$$\frac{du_1}{\sqrt{f(u_1)}} + \frac{du_2}{\sqrt{f(u_2)}} = dt_1, \qquad \frac{u_1\, du_1}{\sqrt{f(u_1)}} + \frac{u_2\, du_2}{\sqrt{f(u_2)}} = dt_2.$$

Functions $u_{1,2}(t_1,t_2)$ have to be found by inversion of the Abel map

$$\int^{u_1}\frac{du_1}{\sqrt{f(u_1)}}+\int^{u_2}\frac{du_2}{\sqrt{f(u_2)}}=t_1+\beta_1\,,$$

$$\int^{u_1}\frac{u_1\,du_1}{\sqrt{f(u_1)}}+\int^{u_1}\frac{u_2\,du_2}{\sqrt{f(u_2)}}=t_2+\beta_2.$$

or

$$\int^{u_1}\omega_1+\int^{u_2}\omega_1=t_1+\beta_1\,,\qquad \int^{u_1}\omega_2+\int^{u_1}\omega_2=t_2+\beta_2.$$

Such equations appear for many integrable systems with two-degrees of freedom : Neumann, Garnier, Clebsch, Rosochatius, Hénon-Heiles systems, Kowalevski top, Goryachev-Chaplygin top etc.

Suppose that transformation of variables

$$\mathcal{B}: \quad (u_1, u_2, p_{u_1}, p_{u_2}) \to (\tilde{u}_1, \tilde{u}_2, \tilde{p}_{u_1}, \tilde{p}_{u_2})$$

preserves Hamilton equations and form of Hamiltonians, i.e. that

$$\tilde{p}_{u_i}^2 = f(\tilde{u}_i), \qquad f(u) = a_6 u^6 + a_5 u^5 + a_4 u^4 + a_3 u^3 + a_2 u^2 + a_1 u + a_0.$$

and

$$\frac{d\tilde{u}_1}{\sqrt{f(\tilde{u}_1)}} + \frac{d\tilde{u}_2}{\sqrt{f(\tilde{u}_2)}} = dt_1, \qquad \frac{\tilde{u}_1\, d\tilde{u}_1}{\sqrt{f(\tilde{u}_1)}} + \frac{\tilde{u}_2\, d\tilde{u}_2}{\sqrt{f(\tilde{u}_2)}} = dt_2.$$

Subtracting these equations from

$$\frac{du_1}{\sqrt{f(u_1)}} + \frac{du_2}{\sqrt{f(u_2)}} = dt_1, \qquad \frac{u_1\, du_1}{\sqrt{f(u_1)}} + \frac{u_2\, du_2}{\sqrt{f(u_2)}} = dt_2$$

one gets Abel differential equations.

Abel equations

$$\omega_1(x_1,y_1)+\omega_1(x_2,y_2)+\omega_1(x_3,y_3)+\omega_1(x_3,y_3)=0\,,$$

$$\omega_2(x_1,y_1)+\omega_2(x_2,y_2)+\omega_2(x_3,y_3)+\omega_2(x_3,y_3)=0\,,$$

where

$$x_{1,2}=u_{1,2},\quad y_{1,2}=p_{u_{1,2}}\,,\qquad x_{3,4}=\tilde{u}_{1,2},\quad y_{3,4}=-\tilde{p}_{u_{1,2}}$$

and $\omega_{1,2}$ form a base of holomorphic differentials

$$\omega_1(x,y)=\frac{dx}{y}\,,\qquad \omega_2(x,y)=\frac{xdx}{y}\,.$$

According to Abel, solutions ot these equations are points of intersection of two plane curves, i.e. solutions of algebraic equations

$$y^2=f(x)=0\,,\qquad h(x,y,t)=y-P(x)=0\,.$$

In our case

$$P(x) = \sqrt{a_6}x^3 + b_2x^2 + b_1x + b_0\,,$$

i.e. there are six points of intersection such that

- four points $(x_1, y_1), \ldots, (x_4, y_4)$ are solutions of Abel equations;
- one point is located at infinity;
- one remaining point with coordinates (λ, μ) is a constant of integration of Abel differential equations.

Eliminating variable y we obtain so-called Abel ploynomial

$$\psi(x) = f(x) - P(x)^2$$

or

$$\psi(x) = \Big(a_{2g+1} - 2\sqrt{a_{2g+2}}\,b_g\Big)(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-\lambda)\,.$$

Dividing roots of the Abel polynomial on two parts one gets

$$(x-x_3)(x-x_4)=\frac{f(x)-P^2(x)}{\left(a_{2g+1}-2\sqrt{a_{2g+2}}\,b_g\right)(x-x_1)\cdots(x-x_2)}\,.$$

and

$$y_{3,4}=P(x=x_{3,4})\,,$$

i.e. desired Bäcklund transformation.

In this particular case the numbers of degrees of freedom is equal to genus of curve and cubic polynomial $P(x)$ may be found using Lagrange interpolation by data

$$P(x)=\sqrt{a_6}x_3+\ldots,\quad y_1=P(x_1)\,,\quad y_2=P(x)\,,\quad \mu=P(\lambda)\,.$$

In this case BT is a shift on Jacobian, but Abel also considered few more complicated cases.

In order to describe algebraic integrals of the Abel equations Jacobi introduce so-called Jacobi polynomials

$$U(x)=\prod_{k=1}^{m}(x-x_k)\,,\qquad V(x)=\sum_{i=1}^{m}y_i\prod_{j\neq i}\frac{z-x_j}{x_i-x_j}\,,$$

and

$$W(x)=\frac{f(x)-V^2(x)}{U(x)}\,,$$

i.e. consider similar transformation of the "Lax matrix"

$$\begin{pmatrix} P(x) & U(x) \\ \psi(x)U^{-1}(x) & -P(x)\end{pmatrix}\to M\begin{pmatrix} V(x) & U(x) \\ W(x) & -V(x)\end{pmatrix}M^{-1}$$

where

$$M=\begin{pmatrix} U(x) & 0 \\ V(x)-P(x) & U(x)\end{pmatrix}$$

Let us consider Hénon-Heiles system with Hamiltonian

$$H_1 = \frac{p_1^2 + p_2^2}{4} - 4aq_2(q_1^2 + 2q_2^2)$$

separable in parabolic coordinates on the plane

$$u_1 = q_2 - \sqrt{q_1^2 + q_2^2}, \qquad u_2 = q_2 + \sqrt{q_1^2 + q_2^2}\,.$$

Their images after BT at $\lambda = 0$ are roots of polynomial

$$(x - \tilde{u}_1)(x - \tilde{u}_2) = x^2 + \left(u_1 + u_2 - \frac{(p_{u_1} - p_{u_2})^2}{a(u_1 - u_2)^2}\right)x$$
$$+ u_1^2 + u_1 u_2 + u_2^2 - \frac{p_{u_1}^2 - p_{u_2}^2}{a(u_1 - u_2)}$$

and

$$\tilde{p}_{1,2} = \tilde{u}_{1,2}^{-1}\,P(\tilde{u}_{1,2})\,, \qquad P(x) = \frac{x(x - u_2)}{u_1 - u_2} p_{u_1} + \frac{x(x - u_1)}{u_2 - u_1} p_{u_2}\,.$$

Substituting these new canonical variables into a pair of separation relations

$$\Phi_i = \left(\tilde{p}_{u_i} - a\,\tilde{u}_i^3\right) - \tilde{H}_1 \pm \sqrt{\tilde{H}_2}\,, \qquad i = 1, 2$$

and solving the resulting equations with respect to $\tilde{H}_{1,2}$, one gets well-known Hamilton function

$$\tilde{H}_1 = \left(\tilde{p}_{u_1} - a\,\tilde{u}_1^3\right) + \left(\tilde{p}_{u_2} - a\,\tilde{u}_2^3\right) = \frac{p_1^2}{4} + \frac{p_2^2}{2} - 2aq_2(3q_1^2 + 8q_2^2)$$

for another integrable case of the Hénon-Heiles system.

Second integral of motion $\tilde{H}_2$ is a polynomial of fourth order in momenta. In the similar manner we can relate well-known Neumann and Chaplygin systems on a sphere.

Jacobi system on ellipsoid

Consider an ellipsoid in $\mathbb{R}^3$

$$\mathbb{E}^2 = \{q \in \mathbb{R}^3, \quad (A^{-1}q, q) = 1\},$$

where $A = \mathrm{diag}(a_1, a_2, a_3)$ and (x, y) - scalar product.

The Hamilton function

$$H == \frac{1}{2}(p, p) + 2\kappa^2 (q, q)$$

and Dirac-Poisson bracket

$$\{q_i, q_j\} = 0, \quad \{y_i, y_j\} = -\frac{q_i p_j - q_j p_i}{a_i a_j (A^{-2} q, q)},$$

$$\{q_i, p_j\} = \delta_{ij} - \frac{q_i q_j}{a_i a_j (A^{-2} q, q)}$$

give rise to Hamiltonian equations of motion.

Jacobi system on ellipsoid

In elliptic coordinates $u_{1,2}$

$$\frac{(\lambda-u_1)(\lambda-u_2)\lambda}{(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)}=1+\frac{q_1^2}{\lambda-a_1}+\frac{q_2^2}{\lambda-a_2}+\frac{q_3^2}{\lambda-a_3}$$

equations of motion

$$\frac{du_1}{dt_1}=\{H_1,u_1\}=\frac{\sqrt{f(u_1)}}{2u_1(u_1-u_2)}\,,\qquad \frac{du_2}{dt_1}=\{H_1,u_2\}=\frac{\sqrt{f(u_2)}}{2u_2(u_2-u_1)}\,,$$

$$\frac{du_1}{dt_2}=\{H_2,u_1\}=\frac{u_2\sqrt{f(u_1)}}{2u_1(u_2-u_1)}\,,\qquad \frac{du_2}{dt_2}=\{H_2,u_2\}=\frac{u_1\sqrt{f(u_2)}}{2u_2(u_1-u_2)}$$

yield "non-standard" quadratures

$$\frac{u_1\,du_1}{\sqrt{f(u_1)}}+\frac{u_2\,du_2}{\sqrt{f(u_2)}}=2dt_2\,,\qquad \frac{u_1^2\,du_1}{\sqrt{f(u_1)}}+\frac{u_2^2\,du_2}{\sqrt{f(u_2)}}=2dt_1\,.$$

Weierstrass transformation

Following to K. Weierstrass,

Über die geodätischen Linien auf dem dreiachsigen Ellipsoid

we can change time $t \to s$, so that equations of motion

$$\frac{du_1}{ds_1} = \frac{\sqrt{f(u_1)}}{2(u_1 - u_2)}, \qquad \frac{du_2}{ds_1} = \frac{\sqrt{f(u_2)}}{2(u_2 - u_1)},$$

$$\frac{du_1}{ds_2} = \frac{u_2\sqrt{f(u_1)}}{2(u_2 - u_1)}, \qquad \frac{du_2}{ds_2} = \frac{u_1\sqrt{f(u_2)}}{2(u_1 - u_2)},$$

yield standard quadratures

$$\frac{du_1}{\sqrt{f(u_1)}} + \frac{du_2}{\sqrt{f(u_2)}} = 2ds_2\,, \quad \frac{u_1\,du_1}{\sqrt{f(u_1)}} + \frac{u_2\,du_2}{\sqrt{f(u_2)}} = 2ds_1$$

on the hyperelliptic curve $\mathcal{C}$ of genus two

$$y^2 = f(x)\,, \qquad f(x) = x(a_1 - x)(a_2 - x)(a_3 - x)(-\kappa^2 x^2 + H_1 x + H_2)\,.$$

$$\mathcal{B}: \qquad (u_1, u_2, p_{u_1}, p_{u_2}) \leftrightarrow (\tilde{u}_1, \tilde{u}_2, \tilde{p}_{u_1}, \tilde{p}_{u_2})$$

one gets two standard Abel differential equations

$$\begin{aligned} &\omega_1(x_1, y_1) + \omega_1(x_2, y_2) + \omega_1(x_3, y_3) + \omega_1(x_4, y_4) = 0\,, \\ &\omega_2(x_1, y_1) + \omega_2(x_2, y_2) + \omega_2(x_3, y_4) + \omega_2(x_4, y_4) = 0\,. \end{aligned}$$

and standard Abel polynomial

$$\psi(x) = \Big(\mathrm{a}_5 - 2\sqrt{\mathrm{a}_6}\, b_2\Big)(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - \lambda)\,.$$

Here

$$\begin{aligned} x_{1,2} &= u_{1,2}\,, \qquad & y_{1,2} &= (a_1 - u_{1,2})(a_2 - u_{1,2})(a_3 - u_{1,2})p_{u_{1,2}} \\ x_{3,4} &= \tilde{u}_{1,2}\,, \qquad & y_{3,4} &= -(a_1 - \tilde{u}_{1,2})(a_2 - \tilde{u}_{1,2})(a_3 - \tilde{u}_{1,2})\tilde{p}_{u_{1,2}} \end{aligned}$$

and

$$\mathrm{a}_6 = \kappa^2\,, \qquad \mathrm{a}_5 = -(a_1 + a_2 + a_3)\kappa^2 - H_1\,.$$

Weierstrass equations of motion

$$\frac{du_1}{ds_1}=\frac{\sqrt{f(u_1)}}{2(u_1-u_2)}\,,\qquad \frac{du_2}{ds_1}=\frac{\sqrt{f(u_2)}}{2(u_2-u_1)}\,,$$

$$\frac{du_1}{ds_2}=\frac{u_2\sqrt{f(u_1)}}{2(u_2-u_1)}\,,\qquad \frac{du_2}{ds_2}=\frac{u_1\sqrt{f(u_2)}}{2(u_1-u_2)}\,,$$

are Hamiltonian w.r.t Poisson bracket

$$\{u_i,p_{u_j}\}_W=u_i\delta_{ij}\,,\qquad \{u_1,u_2\}_W=\{p_{u_1},p_{u_2}\}_W=0$$

and, therefore, our BT preserves form of this bracket

$$\{\tilde{u}_i,\tilde{p}_{u_j}\}_W=\tilde{u}_i\delta_{ij}\,,\qquad \{\tilde{u}_1,\tilde{u}_2\}_W=\{\tilde{p}_{u_1},\tilde{p}_{u_2}\}_W=0$$

and change the form of original canonical bracket.

$$\rho: \qquad (u_1, u_2, p_{u_1}, p_{u_2}) \to (u_1, u_2, u_1 p_{u_1}, u_2 p_{u_2})$$

reduce canonical bracket $\{.,.\}$ to bracket $\{.,.\}_W$.

The images of elliptic coordinates u, p_u after mapping mappings $\rho \circ \mathcal{B}$ are new canonical variables $\hat{u}, \hat{p}_u$ on $T^*\mathbb{E}$.

Substituting new canonical variables into the following separation relations

$$\Phi_i = \Big(\varphi(\hat{u}_i)\hat{p}^2_{u_i} - \kappa^2 \hat{u}_i\Big) - \hat{H}_1 \pm \sqrt{\hat{H}_2}\,, \qquad i = 1, 2$$

we obtain new integrable system on ellipsoid with fourth order integral of motion.

Hamilton function has a natural form

$$\hat{H}_1=\sum_{ij}\hat{\mathrm{g}}_{ij}p_{u_i}p_{u_j}-\frac{2\kappa u_1u_2}{u_1-u_2}(\varphi_1u_2p_{u_1}-\varphi_2u_1p_{u_2})$$

$$+\kappa^2\left(2\alpha_3(u_1+u_2)-\alpha_2u_1u_2+u_1^2u_2^2\right),$$

where metric is equal to

$$\hat{\mathrm{g}}=(u_1-u_2)^{-2}\left(\begin{array}{cc} u_1\varphi_1\eta_1 & -u_1u_2\varphi_1\varphi_2\\ -u_1u_2\varphi_1\varphi_2 & u_2\varphi_2\eta_2\end{array}\right)$$

$$\eta_1=\alpha_3(2u_1-u_2)-u_1(\alpha_2u_2-\alpha_1u_2^2+u_1u_2^2)$$

$$\eta_2=\alpha_3(2u_2-u_1)-u_2(\alpha_2u_1-\alpha_1u_1^2+u_1^2u_2)$$

and

$$\varphi_k=(a_1-u_k)(a_2-u_k)(a_3-u_k),\qquad \alpha_3=a_1a_2a_3,$$

$$\alpha_2=a_1a_2+a_1a_3+a_2a_3,\qquad \alpha_1=a_1+a_2+a_3,$$

The number of degrees of freedom is not equal to the genus of curve

Following to Abel's papers in Crelle's Journal **3**, 4 and works by Jacobi, Weierstrass and Richelot now we consider BT, which can not be realized as a trivial shift of Jacobian.

Take Abel equation on genus two curve

$$\frac{du_1}{\sqrt{f(u_1)}} + \frac{du_2}{\sqrt{f(u_2)}} + \frac{u_3}{\sqrt{f(u_3)}} + \frac{u_4}{\sqrt{f(u_4)}} = 0\,.$$

glue even and odd points $u_1 = u_2$, $u_2 = u_4$ in order to get

$$\frac{du_1}{\sqrt{f(u_1)}} + \frac{du_2}{\sqrt{f(u_2)}} = 0\,,$$

where $f(u) = a_6 u^6 + a_5 u^5 + \dots$

So, for one dimensional Hamiltonian system on genus two curve Abel and Jacobi polynomials have multiple roots and instead of Lagrange interpolation we have to use Hermite interpolation.

Of course, in this case **BT is not a shift of Jacobian.**

The number of degrees of freedom is more then genus of the curve

Let us take integrable system with Hamiltonians

$$H_1=\frac{p_1^2+p_2^2}{4}+\frac{a}{q_1^2}-\frac{2bq_2}{q_1^4}+\frac{c(q_1^2+4q_2^2)}{q_1^6}\,,\qquad a,b,c\in\mathbb{R}\,,$$

$$H_2=-\frac{p_1(p_1q_2-p_2q_1)}{2}-\frac{2aq_2}{q_1^2}+\frac{b(q_1^2+4q_2^2)}{q_1^4}-\frac{4cq_2(q_1^2+2q_2^2)}{q_1^6}\,.$$

separable in parabolic coordinates $u_{1,2}$ and separation relations

$$\Phi(u_i,p_i,H_1,H_2)=\left(u_i^2p_{u_i}\right)^2-(H_1u_i^4+H_2u_i^3-au_i^2-bu_i-c)=0\,,$$

determine an elliptic curve

$$\mathcal{C}:\qquad y^2=f(x)\,,\qquad f(x)=h_1u^4+h_2u^3-au^2-bu-c\,.$$

Genus is less then numbers of degrees of freedom.

We identify variables

$$x_{1,2} = u_{1,2}, \qquad y_{1,2} = u_{1,2}^2 p_{u_{1,2}}$$

with abscissas and ordinates of two points on elliptic curve $\mathcal{C}$ and consider Hamiltonian equations of motion

$$\frac{dx_1}{dt} = \{u_1, H_1\} = \frac{\partial H_1}{\partial p_{u_1}} = \frac{2y_1}{x_1(x_1 - x_2)}$$

$$\frac{dx_2}{dt} = \{u_2, H_1\} = \frac{\partial H_1}{\partial p_{u_2}} = \frac{2y_2}{x_2(x_2 - x_1)},$$

which allow us to obtain Abel quadratures

$$\frac{x_1\,dx_1}{y_1} + \frac{x_2\,dx_2}{y_2} = 0 \quad \frac{x_1^2\,dx_1}{y_1} + \frac{x_2^2\,dx_2}{y_2} = 2dt$$

and similar for the second flow.

Following to Weierstrass we change time $t \to s$ and introduce new equations

$$\frac{dx_1}{ds} = \{u_1, H_1\}_W = \frac{2x_1 y_1}{x_1 - x_2}, \qquad \frac{dx_2}{ds} = \{u_2, H_1\}_W = \frac{2x_2 y_2}{x_2 - x_1}.$$

in order to reduce second Abel quadrature to the standard form

$$\frac{dx_1}{x_1 y_1} + \frac{dx_2}{x_2 y_2} = 0, \quad \frac{dx_1}{y_1} + \frac{dx_2}{y_2} = 2ds_1.$$

These equations are Hamiltonian w.r.t. Poisson bracket

$$\{u_i, p_{u_j}\}_W = u_i^2 \delta_{ij}, \qquad \{u_1, u_2\}_W = \{p_{u_1}, p_{u_2}\}_W = 0,$$

which is compatible with original canonical Poisson brackets.

Suppose that transformation of variables

$$\mathcal{B}: \qquad (u_1, u_2, p_{u_1}, p_{u_2}) \leftrightarrow (\tilde{u}_1, \tilde{u}_2, \tilde{p}_{u_1}, \tilde{p}_{u_2})$$

preserves form of Hamilton equations after change of time and form of the stationary Hamilton-Jacobi equations, which are independent on time.

By definition variables $x_{1,2}, y_{1,2}$ and

$$x_{3,4} = \tilde{u}_{1,2}\,, \qquad y_{3,4} = -\tilde{u}_{1,2}^2 \tilde{p}_{u_{1,2}}$$

satisfy to Abel differential equation on elliptic curve

$$\frac{dx_1}{y_1} + \frac{dx_2}{y_2} + \frac{dx_3}{y_3} + \frac{dx_4}{y_4} = 0$$

which relates all four points of intersection.

In this case Abel polynomial is equal to

$$\psi(x) = f(x) - P(x)^2 = A(x - x_1)(x - x_2)(x - x_3)(x - x_4)\,,$$

where $A = H_1 - b_2^2$ and b_2 is a leading coefficients of polynomial

$$P(x) = b_2 x^2 + b_1 x + b_0\,,$$

such that

$$y_{1,2} = P(x_{1,2})\,, \qquad y_{3,4} = P(x_{3,4})$$

Following to Abel we can uniquely determine this polynomial

$$P(x) = x\left(\frac{(x - x_2)y_1}{x_1(x_1 - x_2)} + \frac{(x - x_1)y_2}{x_2(x_2 - x_1)}\right)$$

using these data and complete integral of Abel equation

$$\begin{vmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \\ x_4^2 & x_4 & 1 & y_4 \end{vmatrix} = 0\,.$$

Using composition of the BT and the Poisson map

$$\rho: \qquad (u_1, u_2, p_{u_1}, p_{u_2}) \to (u_1, u_2, u_1^2 p_{u_1}, u_2^2 p_{u_2}),$$

which reduces canonical Poisson bracket $\{.,.\}$ to $\{.,.\}_W$ we easily obtain new canonical variables $\hat{u}_{1,2}$, which are solutions of equation

$$\left(\rho(H_1) - \left(\frac{u_1^3 p_{u_1} - u_2^3 p_{u_2}}{u_1 - u_2}\right)^2\right) x^2 - \frac{b u_1 u_2 + c(u_1 + u_2)}{u_1^2 u_2^2}\, x - \frac{c}{u_1 u_2} = 0$$

and momenta

$$\hat{p}_{u_{1,2}} = -\hat{u}_{1,2}^{-4}\, \rho(P)\Big|_{x=\hat{u}_{1,2}}.$$

If we substitute these variables into the separated relations

$$\Phi_i(\hat{u}_i,\hat{p}_{u_i}) = \left(\hat{u}_i^4\hat{p}_{u_i}^2 + \frac{a}{\hat{u}_i^2} + \frac{b}{\hat{u}_i^3} + \frac{c}{\hat{u}_i^4}\right) - \hat{H}_1 \pm \sqrt{\hat{H}_2} = 0\,,$$

and solve these relations with respect to $\hat{H}_{1,2}$ one gets new integrable system with "natural" Hamilton function

$$\hat{H}_1 = \frac{\big(bu_1u_2+c(3u_1+u_2)\big)u_1^4p_{u_1}^2}{c(u_1-u_2)} + \frac{\big(bu_1u_2+c(u_1+3u_2)\big)u_2^4p_{u_2}^2}{c(u_1-u_2)}$$

$$- \frac{(bu_1+c)(au_1^2+bu_1+c)}{cu_1^4} - \frac{(bu_2+c)(au_2^2+bu_2+c)}{cu_2^4}$$

$$- \frac{4ac+b^2}{cu_1u_2} - \frac{5b(u_1+u_2)}{u_1^2u_2^2} - \frac{4c(u_1^2+u_1u_2+u_2^2)}{u_1^3u_2^3}$$

where "metric" and "potential" depend on the same parameters a,b,c and $\hat{H}_2$ are polynomial of sixth order in momenta.

Starting with other separation relations

$$
\begin{aligned}
(u^2p_u)^2 - (H_1u^4 + H_2u^3 + au^2 + bu + c) = 0\,,\\
(u^2p_u)^2 - (au^4 + H_1u^3 + H_2u^2 + bu + c) = 0\,,\\
(u^2p_u)^2 - (H_1u^4 + au^3 + bu^2 + cu + H_2) = 0\,,\\
\ldots
\end{aligned}
$$

we can prove that following metrics on the plane

$$
\hat{g}_{km} = \begin{pmatrix} \dfrac{(ku_1+u_2)u_1^m}{u_1-u_2} & 0 \\ 0 & \dfrac{(ku_2+u_1)u_2^m}{u_2-u_1} \end{pmatrix}
$$

give rise to integrable geodesic flows if:

$$
\begin{aligned}
&m=1, \quad k=2; \qquad m=3\,, \quad k=\pm 1, 3, \frac{1}{2}\,,\\
&m=4\,, \quad k=\pm 1, \pm 3, -\frac{3}{5}, -\frac{1}{7}, \frac{1}{5}, \frac{1}{2}\,.
\end{aligned}
$$

Conclusion:

There is algebraic construction of the auto Bäcklund transformations for the Hamilton-Jacobi equations solvable by inversion of the Abel quadratures on the hyperelliptic and non-hyperelliptic curves.

In fact, knowing quadratures we immediately obtain Abel's differential equations and equivalence relation on the symmetric product of these curves, which can be identified with auto Bäcklund transformation.

Auto Bäcklund transformations depend on arbitrary fixed roots λ_i of the Abel polynomial if this polynomial has such roots.

Using these transformations we can get a lot of new integrable systems with integrals of motion of third, fourth and sixth order in momenta.

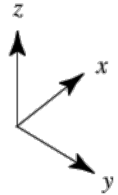
Integrals of motion for the Goryachev system

$$
\begin{aligned}
H_1 &= J_1^2 + J_2^2 + \frac{4}{3}J_3^2 + \frac{a\gamma_1}{\gamma_3^{2/3}} + \frac{b}{\gamma_3^{2/3}}\,,\\
H_2 &= -\frac{2J_3}{3}\left(J_1^2 + J_2^2 + \frac{8}{9}J_3^2 + \frac{b}{\gamma_3^{2/3}}\right) + \frac{a(3\gamma_3 J_1 - 2\gamma_1 J_3)}{3\gamma_3^{2/3}}\,.
\end{aligned}
$$

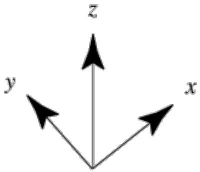
Integrable map

$$
(x_1, x_2, x_3, J_1, J_2, J_3) \to (x_1, -x_2, x_3, J_1, -J_2, J_3)
$$

describes symmetry of left-handed and right-handed coordinate systems



left-handed coordinate system



right-handed coordinate system

In order to describe this trivial Bäcklund transformation we introduce variables of separation

$$q_1+q_2=\frac{\gamma_3^{4/3}J_3}{1-\gamma_3^2}+\frac{\mathrm{i}(J_1\gamma_2-\gamma_1J_2)\gamma_3^{1/3}}{1-\gamma_3^2},\qquad q_1q_2=\frac{a}{2(\gamma_1+\mathrm{i}\gamma_2)},$$

$$p_{1,2}=\frac{3\mathrm{i}\gamma_3^{2/3}}{2}+\frac{\mathrm{i}J_3}{q_{1,2}}.$$

which are subject to the algebraic relation

$$f(x,y)=x^4-bx^2+(y^3-h_1y+h_2)x+\frac{a^2}{4}=0.$$

The corresponding algebraic curve is a trigonal curve of genus 3.

Using a base of holomorphic differentials

$$\omega_1(x,y)=\frac{dx}{\partial f/\partial y}\,,\qquad \omega_2(x,y)=\frac{xdx}{\partial f/\partial y}\,,\qquad \omega_3(x,y)=\frac{ydx}{\partial f/\partial y}$$

we present Hamilton equations of motion,

$$\omega_1(x_1,y_1)+\omega_1(x_2,y_2)=dt_1\,,\qquad \omega_3(x_1,y_1)+\omega_3(x_2,y_2)=dt_2\,.$$

where $q_{1,2}=x_{1,2}$ and $p_{1,2}=y_{1,2}$. Consider transformation of variables

$$\mathcal{B}:\qquad (q_1,q_2,p_1,p_2)\leftrightarrow(\tilde{q}_1,\tilde{q}_2,\tilde{p}_1,\tilde{p}_2)\,,$$

such that new coordinates $\tilde{q}_1=x_3$ and $\tilde{q}_2=x_4$ satisfy

$$-\omega_1(x_3,y_3)-\omega_1(x_4,y_4)=dt_1\,,\qquad -\omega_3(x_3,y_3)-\omega_3(x_4,y_4)=dt_2\,.$$

$$\begin{aligned}\omega_1(x_1,y_1)+\omega_1(x_2,y_2)+\omega_1(x_3,y_3)+\omega_1(x_4,y_4) &= 0\,,\\ \omega_3(x_1,y_1)+\omega_3(x_2,y_2)+\omega_3(x_3,y_3)+\omega_3(x_4,y_4) &= 0\,,\end{aligned}$$

are the point of intersection of trigonal curve $f(x,y)=0$ with a family of lines

$$h(x,y,t)=y-P(x)=0,\qquad P(x)=\frac{x_1y_2-x_2y_1}{x_1-x_2}x+\frac{y_1-y_2}{x_1-x_2}\,.$$

Abel polynomial ψ gives rise to desired symmetry

$$\begin{aligned}(x-x_3)(x-x_4) &= x^2+\frac{(x_1+x_2)(x_1-x_2)^3+(y_1-y_2)^2(x_1y_1+2x_1y_2-2x_2y_1-x_2y_2)}{\Big((x_1-x_2)^2+(y_1-y_2)(y_1-y_2+x_1-x_2)\Big)(y_1-y_2+x_1-x_2)}\,x\\ &+ \frac{a^2(x_1-x_2)^3}{4x_1x_2\Big((x_1-x_2)^2+(y_1-y_2)(y_1-y_2+x_1-x_2)\Big)(y_1-y_2+x_1-x_2)}\,.\end{aligned}$$

In order to get LHS to RHS symmetry we eliminate variables y from equations of curves

$$g(x, y) = 0\,, \qquad h(x, y, t) = 0.$$

and this map is not a shift on Jacobian of trigonal curve.

In order to get more complicated Bäcklund transformation we have to eliminate variable x from the equations of trigonal curve $f(x, y) = 0$ and equations of a family of lines $h(x, y, t)$.

In this case Abel polynomial $\psi(y)$ has multiple roots and we have to apply more complicated Lagrange interpolation formulae for polynomial $P(y)$.