

"Polynomial and Elliptic Algebras, Heisenberg group and Cremona transformations"

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Russian Federation, Chechen Republic, Kezenoi-Am, 2.11.2016

This talk partly based on my joint papers with:

Sasha Odesskii (Brok Univ. Canada) :

- Theoret. and Math. Phys. 133 (2002), no. 1, 1321-1337
- Integrable systems associated with elliptic algebras. Quantum groups, 8105, IRMA Lect. Math.Theor. Phys., 12, Eur. Math. Soc., Zurich, 2008

and Serge Tagné Pelap ("Credit Suisse", Zurich) and Giovanni Ortenzi (Univ. Milan, Bicocca):

- On the Heisenberg invariance and the elliptic Poisson tensors. Lett. Math. Phys. 96 (2011), no. 1-3, 263-284
- Integer Solutions of Integral Inequalities and H -Invariant Jacobian Poisson Structures, Adv. in Math. Phys. Vol. 2011 (2011), Article ID 252186, doi:10.1155/2011/252186
- paper(s?) in progress...

Plan

- 1 Introduction
 - Poisson brackets and structures per se
 - Poisson algebras associated to elliptic curves.
- 2 Heisenberg invariance, Poisson structures on Moduli spaces, Odesskii-Feigin-Polishchuk
- 3 Unimodularity
- 4 Cremona transformations and Poisson morphisms of \mathbb{P}^4
- 5 $n = 7$, perspectives...

Poisson brackets

- $\{f, g\} = -\{g, f\}$
- $\{f, gh\} = -\{f, g\}h + \{f, h\}g, f$
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

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A **Poisson structure** on a manifold M (smooth or algebraic) is given by a bivector: antisymmetric tensor field $\pi \in \Lambda^2(TM)$ defining on the corresponded algebra of functions on M a structure of (infinite dimensional) Lie algebra by means of the **Poisson brackets**

$$\{f, g\} = \langle \pi, df \wedge dg \rangle.$$

The Jacobi identity for this brackets is equivalent to an analogue of (classical) Yang-Baxter equation namely to the "Poisson Master Equation": $[\pi, \pi] = 0$, where the brackets $[,] : \Lambda^p(TM) \times \Lambda^q(TM) \mapsto \Lambda^{p+q-1}(TM)$ are the only Lie super-algebra structure on $\Lambda^\cdot(TM)$.

Analogue of the **Darboux theorem** (A. Weinstein) \Rightarrow local description of any Poisson manifold.

There are coordinates $(q_1, \dots, q_l, p_1, \dots, p_l, x_1, \dots, x_k)$ near any point $m \in M$ such that π reads as

$$\pi = \sum_{i=1}^l \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i>j}^k f_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

with $f(m) = 0$.

Casimirs

A function $F \in Fun(M)$ is a **Casimir** of the Poisson structure π if $\{F, G\} = 0$ for all functions $G \in Fun(M)$.

If the rank of the structure is constant in this neighborhood of m (m is called a regular point) then the Casimirs in the neighborhood are the functions depending only on x_1, \dots, x_k and Poisson manifold admits a **foliation by symplectic leaves**, i.e. is a unification of submanifolds

$$x_1 = c_1, \dots, x_k = c_k$$

and c_i are constants such that π is non-degenerate on each of them. In general the dimension of the leaves is constant only on the open dense

complex Poisson manifold

- $\pi = \sum \pi^{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} \in H^0(M, \Lambda^2 T)$
- $[\pi, \pi] = 0 \in H^0(M, \Lambda^3 T)$
- f, g local holomorphic functions, then $\{f, g\} = \langle \pi, df \wedge dg \rangle$.

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- $\pi \in H^0(M, \Lambda^2 T) = H^0(M, K^*)$
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case of $M = \mathbb{P}^2$

- $M = \mathbb{P}^2, K^* = \mathcal{O}(3)$.
- π vanishes on an elliptic curve.
- $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ vanishes to order 3 on the line at infinity.

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Poisson threefolds

- $\pi \in H^0(M, \Lambda^2 T), [\pi, \pi] = 0.$
- $\Lambda^2 T \simeq T^* \otimes K^*, \pi \rightarrow \theta$
- $[\pi, \pi] = 0 \Leftrightarrow \theta \wedge d\theta = 0.$
- singular holomorphic foliation
- symplectic leaves: $\pi \neq 0, \theta = 0.$
- Poisson surfaces intersection in $\pi = 0.$

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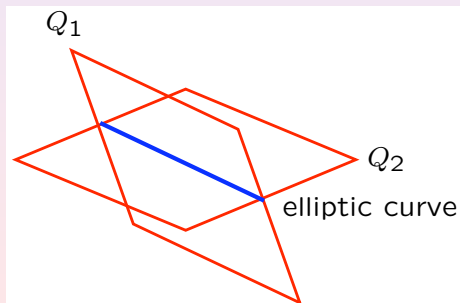
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case of $M = \mathbb{P}^3$

- $M = \mathbb{P}^3, K^* = \mathcal{O}(4)$.
- $s_1, s_2 \in H^0(\mathbb{P}^3, \mathcal{O}(2))$
- $s_i = 0$: quadric surfaces $Q_i \subset \mathbb{P}^3$.



Jacobi-Nambu-Poisson

Let us consider $n - 2$ polynomials Q_i in \mathbb{C}^n with coordinates $x_i, i = 1, \dots, n$. For any polynomial $\lambda \in \mathbb{C}[x_1, \dots, x_n]$ we can define a bilinear differential operation

$$\{, \} : \mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[x_1, \dots, x_n] \mapsto \mathbb{C}[x_1, \dots, x_n]$$

by the formula

$$\{f, g\} = \lambda \frac{df \wedge dg \wedge dQ_1 \wedge \dots \wedge dQ_{n-2}}{dx_1 \wedge dx_2 \wedge \dots \wedge dx_n}, \quad f, g \in \mathbb{C}[x_1, \dots, x_n]. \quad (1)$$

Sklyanin algebra

The case $n = 4$ in (1) corresponds to the classical (generalized) Sklyanin quadratic Poisson algebra. The very Sklyanin algebra is associated with the following two quadrics in \mathbb{C}^4 :

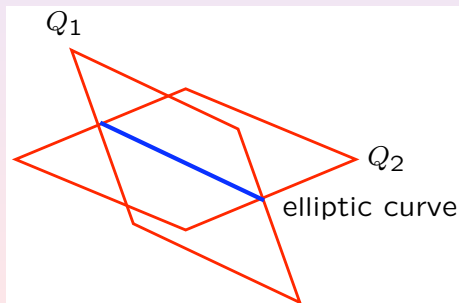
$$Q_1 = x_1^2 + x_2^2 + x_3^2, \quad (2)$$

$$Q_2 = x_4^2 + J_1 x_1^2 + J_2 x_2^2 + J_3 x_3^2. \quad (3)$$

$$J_1 + J_2 + J_3 + J_1 J_2 J_3 \neq 0.$$

The Poisson brackets (1) with $\lambda = 1$ between the affine coordinates looks as follows

$$\{x_i, x_j\} = (-1)^{i+j} \det \left(\frac{\partial Q_k}{\partial x_l} \right), l \neq i, j, i > j. \quad (4)$$



Artin-Tate-Sklyanin elliptic Poisson algebra

- $\mathcal{E} \subset \mathbb{C}P^2$ normal elliptic curve
- L line bundle degree 3
-







$$\mathcal{E} := P(x_1, x_2, x_3) = 1/3(x_1^3 + x_2^3 + x_3^3) + kx_1x_2x_3 = 0, \quad (5)$$

- then

$$\{x_1, x_2\} = kx_1x_2 + x_3^2$$

$$\{x_2, x_3\} = kx_2x_3 + x_1^2$$

$$\{x_3, x_1\} = kx_3x_1 + x_2^2$$

N	Kodaira cycles	Kodaira fibers
$N = 1$	$l_1 : x_2^2 x_3 = x_1^3 + x_1^2 x_3$ 	$II : x_2^2 x_3 = x_1^3$ 
$N = 2$	$l_2 : x_3^3 = x_1 x_2 x_3$ 	$III : x_2^2 x_3 = x_1^2 x_2$ 
$N = 3$	$l_3 : x_1 x_2 x_3 = 0$ 	$IV : x_1 x_2^2 = x_1^2 x_2$ 

A wide class of the polynomial Poisson algebras arises as a quasi-classical limit $q_{n,k}(\mathcal{E})$ of the associative quadratic algebras $Q_{n,k}(\mathcal{E}, \eta)$. Here \mathcal{E} is an elliptic curve and n, k are integer numbers without common divisors, such that $1 \leq k < n$ while η is a complex number and $Q_{n,k}(\mathcal{E}, 0) = \mathbb{C}[x_1, \dots, x_n]$.

Feigin-Odesskii-Sklyanin algebras

Let $\mathcal{E} = \mathbb{C}/\Gamma$ be an elliptic curve defined by a lattice $\Gamma = \mathbb{Z} \oplus \tau\mathbb{Z}$, $\tau \in \mathbb{C}$, $\Im\tau > 0$. The algebra $Q_{n,k}(\mathcal{E}, \eta)$ has generators x_i , $i \in \mathbb{Z}/n\mathbb{Z}$ subjected to the relations

$$\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{j-i-r}(-\eta)\theta_{kr}(\eta)} x_{j-r} x_{i+r} = 0$$

and have the following properties:

Basic properties

- $Q_{n,k}(\mathcal{E}, \eta) = \mathbb{C} \oplus Q_1 \oplus Q_2 \oplus \dots$ such that $Q_\alpha * Q_\beta = Q_{\alpha+\beta}$, here $*$ denotes the algebra multiplication. The algebras $Q_{n,k}(\mathcal{E}, \eta)$ are \mathbb{Z} -graded;
- The Hilbert function of $Q_{n,k}(\mathcal{E}, \eta)$ is $\sum_{\alpha \geq 0} \dim Q_\alpha t^\alpha = \frac{1}{(1-t)^n}$.
- $Q_{n,k}(\mathcal{E}, \eta) \simeq Q_{n,k'}(\mathcal{E}, \eta)$, if $kk' \equiv 1 \pmod{n}$;
- The maps $x_i \mapsto x_{i+1}$ et $x_i \mapsto \varepsilon^i x_i$, where $\varepsilon^n = 1$, define automorphisms of the algebra $Q_{n,k}(\mathcal{E}, \eta)$;
- We can consider the algebra $Q_{n,k}(\mathcal{E}, \eta)$ for fixed \mathcal{E} as a flat deformation of the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$.

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$Q_{2m,1}(\mathcal{E}, \eta)$ as an ACIS

A. Odesskii and V.R prove in 2004 the following

Theorem

The elliptic algebra $Q_{2m,1}(\mathcal{E}, \eta)$ has m commuting elements of degree m .

Let $q_{n,k}(\mathcal{E})$ be the correspondent Poisson algebra. The algebra $q_{n,k}(\mathcal{E})$ has $l = \gcd(n, k + 1)$ Casimirs. Let us denote them by $P_\alpha, \alpha \in \mathbb{Z}/l\mathbb{Z}$. Their degrees $\deg P_\alpha$ are equal to n/l .

Quintic elliptic Poisson algebra

Let us consider the algebra $q_{5,1}(\mathcal{E})$:

Example

We have the polynomial ring with 5 generators $x_i, i \in \mathbb{Z}/5\mathbb{Z}$
enabled with the following Poisson bracket:

$$\begin{aligned} \{x_i, x_{i+1}\}_{5,1} &= \left(-\frac{3}{5} k^2 + \frac{1}{5k^3}\right) x_i x_{i+1} - 2 \frac{x_{i+4} x_{i+2}}{k} + \frac{x_{i+3}^2}{k^2} \\ \{x_i, x_{i+2}\}_{5,1} &= \left(-\frac{1}{5} k^2 - \frac{3}{5k^3}\right) x_{i+2} x_i + 2 x_{i+3} x_{i+4} - k x_{i+1}^2 \end{aligned} \quad (6)$$

Here $i \in \mathbb{Z}/5\mathbb{Z}$ and $k \in \mathbb{C}$ is a parameter of the curve $\mathcal{E}_\tau = \mathbb{C}/\Gamma$,
i.e. some function of τ .

Casimir of degree 5

The center $Z(q_{5,1}(\mathcal{E}))$ is generated by the polynomial

$$\begin{aligned} P_{5,1} = & x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + \\ & (1/k^4 + 3k) (x_0^3 x_1 x_4 + x_1^3 x_0 x_2 + x_2^3 x_1 x_3 + x_3^3 x_2 x_4 + x_2^3 x_0 x_3) + \\ & + (-k^4 + 3/k) (x_0^3 x_2 x_3 + x_1^3 x_3 x_4 + x_2^3 x_0 x_4 + x_3^3 x_1 x_0 + x_4^3 x_1 x_2) + \\ & + (2k^2 - 1/k^3) (x_0 x_1^2 x_4^2 + x_1 x_2^2 x_0^2 + x_2 x_0^2 x_4^2 + x_3 x_1^2 x_0^2 + x_4 x_1^2 x_2^2) + \\ & + (k^3 + 2/k^2) (x_0 x_2^2 x_3^2 + x_1 x_3^2 x_4^2 + x_2 x_0^2 x_4^2 + x_3 x_1^2 x_0^2 + x_4 x_1^2 x_2^2) + \\ & + (k^5 - 16 - 1/k^5) x_0 x_1 x_2 x_3 x_4. \end{aligned}$$

Second elliptic Poisson structure for $n = 5$

It follows from the description of Odesskii-Feigin that there are two essentially different elliptic algebras with 5 generators: $Q_{5,1}(\mathcal{E}, \eta)$ and $Q_{5,2}(\mathcal{E}, \eta')$. The corresponding Poisson counter-part of the latter is $q_{5,2}(\mathcal{E})$:

Example

$$\begin{aligned} \{y_i, y_{i+1}\}_{5,2} &= \left(\frac{2}{5}\lambda^2 + \frac{1}{5\lambda^3}\right) y_i y_{i+1} + \lambda y_{i+4} y_{i+2} - \frac{y_{i+3}^2}{\lambda} \\ \{y_i, y_{i+2}\}_{5,2} &= \left(-\frac{1}{5}\lambda^2 + \frac{2}{5\lambda^3}\right) y_{i+2} y_i - \frac{y_{i+3} y_{i+4}}{\lambda^2} + y_{i+1}^2 \end{aligned} \quad (7)$$

where $i \in \mathbb{Z}_5$. The center $Z(q_{5,2}(\mathcal{E})) = \mathbb{C}[P_{5,2}]$.

Heisenberg group

Consider an n -dimensional vector space V and fix a base v_0, \dots, v_{n-1} of V then the

Heisenberg group of level n in the Schrödinger representation is the subgroup $H_n \subset GL(V)$ generated by the operators

$$\sigma : (v_i) \rightarrow v_{i-1}; \quad \tau : v_i \rightarrow \varepsilon_i v_i, \quad (\varepsilon_i)^n = 1, \quad 0 \leq i \leq n-1.$$

This group has order n^3 and is a central extension

$$1 \rightarrow \mathbb{U}_n \rightarrow H_n \rightarrow \mathbb{Z}_n \times \mathbb{Z}_n \rightarrow 1,$$

where \mathbb{U}_n is the group of n -th roots of unity.

Definition

Let D_\bullet be the map : $D_\bullet := \star^{-1} \circ d \circ \star : \mathcal{X}^\bullet(\mathcal{R}) \longrightarrow \mathcal{X}^{\bullet-1}(\mathcal{R})$, where d is the de Rham differential. **The modular class of a Poisson tensor** π on \mathcal{R} is the class of $D_2(\pi)$ in the first Poisson cohomological group $PH^1(\mathcal{R}, \pi)$, associated to π . A Poisson tensor is said to be **unimodular** if its modular class is trivial.

It is known that all Jacobian Poisson structures are unimodular. Therefore H -invariant Poisson structure in dimension 3 and 4 are unimodular. What happens in higher dimension?

Unimodularity theorem

Theorem

Every H -invariant quadratic Poisson structure is unimodular.

Corollary

The Poisson algebras $q_{n,k}(\mathcal{E})$ are unimodular.

CY-manifolds: reminder

A **Calabi - Yau** n -fold or **Calabi - Yau manifold** of dimension n is sometimes defined as a compact n -dimensional Kähler manifold M satisfying one of the following equivalent conditions:

- The canonical bundle of M is trivial.
- M has a holomorphic n -form that vanishes nowhere.
- The structure group of M can be reduced from $U(n)$ to $SU(n)$.
- M has a Kähler metric with global holonomy contained in $SU(n)$.

There are many other inequivalent definitions of Calabi - Yau manifolds.

CY-algebras: reminder

A **Calabi - Yau algebras** play a role of noncommutative analogues of coordinate rings of CY manifolds.

Definition

An algebra A is said to be a **Calabi - Yau algebra** of dimension d if it is homologically smooth and

$$HH^k(A, A \otimes A) = \begin{cases} A & \text{if } k = d \\ 0 & \text{if } k \neq d \end{cases} \quad (8)$$

(V. Ginzburg, 2005)

CY-algebras: quantization-1

Let X be a smooth affine variety over \mathbb{C} with a trivial canonical class and with a Poisson structure π_1 on its smooth function algebra.

M. Kontsevich:

There is a bijection between the set of equivalence classes of star-products $[*_h]$ of π_1 and the set of equivalence classes $[\pi_h]$ of Poisson tensors

$$\pi_h = 0 + \pi_1 h + \dots + \pi_n h^n + \dots$$

on the commutative $\mathbb{C}[[h]]$ -algebra $A[[h]]$.

CY-algebras: quantization-2

Dolgushev :

Theorem

*$(A[[\hbar]], *_\hbar)$ is a CY algebra if and only if π_\hbar is unimodular Poisson structure.*

It is obvious to observe that the canonical quantization (those which the star-product equivalence class corresponds to the equivalence class of a Poisson tensor $\pi_\hbar = 0 + \pi_1 \hbar$) is a CY algebra if π_1 is a unimodular Poisson structure.

CY-algebras: quantization-3

Corollary

The canonical quantization of an H -invariant quadratic Poisson structure is a CY - algebra.

Conjecture: *The elliptic Sklyanin - Odesskii - Feigin Poisson algebras are CY - algebras.*

The conjecture is true for $n = 3, 4$ (Van den Bergh, Bocklandt et al.)

The Heisenberg group action provides the automorphisms of the Sklyanin algebra which are compatible with the grading and defines also an action on the "quasiclassical" limit of the Sklyanin algebras $q_{n,k}(\mathcal{E})$ - the elliptic quadratic Poisson structures on \mathbb{P}^{n-1} which are identified with Poisson structures on some moduli spaces of the degree n and rank $k + 1$ vector bundles with parabolic structure (= the flag $0 \subset V \subset \mathbb{C}^{k+1}$ on the elliptic curve \mathcal{E})

Odesskii-Feigin description-1

Odesskii-Feigin(1995-2000):

Let $\mathcal{M}_{n,k}(\mathcal{E}) = \mathcal{M}(\xi_{0,1}, \xi_{n,k})$ be the moduli space of $k + 1$ -dimensional bundles on the elliptic curve \mathcal{E} with 1-dimensional sub-bundle. $\xi_{0,1} = \mathcal{O}_{\mathcal{E}}$, $\xi_{n,k}$ - indecomposable bundle of degree n and rank k . This moduli space is a space of exact sequences:

$$0 \rightarrow \xi_{0,1} \rightarrow F \rightarrow \xi_{n,k} \rightarrow 0$$

up to an isomorphism.

Odesskii-Feigin description-2

Theorem

$$\mathcal{M}_{n,k}(\mathcal{E}) \cong \mathbb{P}\text{Ext}^1(\xi_{n,k}; \xi_{0,1}) \cong \mathbb{C}P^{n-1}.$$

The Poisson structure $q_{n,k}(\mathcal{E})$ in the "classical limit" ($\eta \rightarrow 0$) of $Q_{n,k}(\mathcal{E}, \eta)$ is a homogeneous quadratic on \mathbb{C}^n and define a Poisson structure on $\mathbb{C}P^{n-1}$ which coincides with the intrinsic Poisson structure on the moduli space of parabolic bundles $\mathcal{M}_{n,k}(\mathcal{E})$.

Polishchuk description-1

A. Polishchuk(1999-2000):

There exists a natural Poisson structure on the moduli space of triples (E_1, E_2, Φ) of stable vector bundles over \mathcal{E} with fixed ranks and degrees, where $\Phi : E_2 \rightarrow E_1$ a homomorphism. For $E_2 = \mathcal{O}_{\mathcal{E}}$ and $E_1 = E$ this structure is exactly the Odesskii-Feigin structure on $\mathbb{P}Ext^1(E, \mathcal{O}_{\mathcal{E}})$.

Polishchuk description-2

Theorem

Let $\mathcal{M}_{n,k}(\mathcal{E}) \cong \mathbb{P}\text{Ext}^1(E, \mathcal{O}_{\mathcal{E}})$ where E is a stable bundle with fixed determinant $\mathcal{O}(nx_0)$ of rank k , $(n, k) = 1$. Suppose in addition that $(n+1, k) = 1$. Then there is a birational transformation (compatible with Poisson structures)

$$\mathcal{M}_{n,k}(\mathcal{E}) \rightarrow \mathcal{M}_{n,\phi(k):=-(k+1)^{-1}}(\mathcal{E}) \cong \mathbb{P}H^0(F),$$

where F is a stable vector bundle of degree n and rank $k+1$. Moreover, the composition

$$\mathcal{M}_{n,k}(\mathcal{E}) \rightarrow \mathcal{M}_{n,\phi(k)}(\mathcal{E}) \rightarrow \mathcal{M}_{n,\phi^2(k)}(\mathcal{E}) \rightarrow \mathcal{M}_{n,k}(\mathcal{E})$$

is the identity.

Odesskii-Feigin "quantum" homomorphisms for 5-generator algebras

Let $Q_{5,1}(\mathcal{E}, \eta)$ and $Q_{5,2}(\mathcal{E}, \eta)$ be "quantum" elliptic Sklyanin algebras corresponded to $q_{5,1}(\mathcal{E})$ and to $q_{5,2}(\mathcal{E})$.

Example

- The algebra $Q_{5,2}(\mathcal{E}, \eta)$ is a subalgebra in $Q_{5,1}(\mathcal{E}, \eta)$ generated by 5 elements with 10 quadratic relations.
- In its turn, the algebra $Q_{5,1}(\mathcal{E}, \eta)$ is a subalgebra in $Q_{5,2}(\mathcal{E}, \eta)$.
- The compositions of embeddings
 $Q_{5,1}(\mathcal{E}, \eta) \rightarrow Q_{5,2}(\mathcal{E}, \eta) \rightarrow Q_{5,1}(\mathcal{E}, \eta)$ transforms the generators $x_i \rightarrow P_{5,1}x_i$ and
 $Q_{5,2}(\mathcal{E}, \eta) \rightarrow Q_{5,1}(\mathcal{E}, \eta) \rightarrow Q_{5,2}(\mathcal{E}, \eta)$ transforms the generators $y_i \rightarrow P_{5,2}y_i$.

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General (naive) definitions

Consider $n + 1$ homogeneous polynomial functions φ_i in $\mathbb{C}[x_0, \dots, x_n]$ of the same degree which are non identically zero. One can associated the rational map:

$$\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^n, [x_0 : \dots : x_n] \mapsto [\varphi_0([x_0, \dots, x_n]) : \dots : \varphi_n([x_0, \dots, x_n])].$$

The family of polynomial φ_i or φ is called a **birational transformation** of \mathbb{P}^n if there exists a rational map $\psi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ such that $\psi \circ \varphi$ is the identity. A birational transformation is also called a **Cremona transformation**.

Let $(\lambda : \mu) \in \mathbb{P}^1$ such that $k = \lambda/\mu$ and $\mathcal{E}_{\lambda,\mu}$ is given by the set of the **quadrics**

$$C_i^0(x) = \lambda\mu x_i^2 - \lambda^2 x_{i+2}x_{i+3} + \mu^2 x_{i+1}x_{i+4}, \quad i \in \mathbb{Z}_5, \quad (9)$$

(These quadrics are 4x4 Pfaffians of the Klein **syzygy** 5x5 skew-symmetric matrix of linear forms.)

The elliptic quintic scroll $Q_{\lambda,\mu}(z)$ is given by the set of cubics

$$Q_i^0(z) = \lambda^2 \mu^2 z_i^3 + \lambda^3 \mu (z_{i+1}^2 z_{i+3} + z_{i+2} z_{i+4}^2) - \lambda \mu^3 (z_{i+1} z_{i+2}^2 + z_{i+3}^2 z_{i+4}) - \lambda^4 z_i z_{i+1} z_{i+4} - \mu^4 z_i z_{i+2} z_{i+3}, \quad i \in \mathbb{Z}_5. \quad (10)$$

The transformation $v_{\pm} : \mathbb{P}^4(x) \mapsto \mathbb{P}^4(z)$ is given in coordinates by

$$v_+ : z_i \rightarrow x_{i+2} x_{i+4}^2 - x_{i+1}^2 x_{i+3} \text{ and by}$$

$$v_- : z_i \rightarrow x_{i+1} x_{i+2}^2 - x_{i+3}^2 x_{i+4}.$$

The incidence variety $I_{\lambda,\mu}$ is the elliptic scroll over the curve $\mathcal{E}_{\lambda,\mu}$ which is transformed by the Cremona transformation to the scroll $S \subset \mathbb{P}^4(w)$.

Theorem

- The quadro-cubic Cremona transformations (9) and (10) are Poisson morphisms of \mathbb{P}^4 which transform $q_{5,1}(\mathcal{E})$ to $q_{5,2}(\mathcal{E})$ and vice versa.
- These Cremona transformations are "quasi-classical limits" of Odesskii-Feigin "quantum" homomorphisms $Q_{5,1}(\mathcal{E}, \eta) \rightarrow Q_{5,2}(\mathcal{E}, \eta)$ and vice versa.
- The Casimir quintics $P_{5,1,2}$ are given by Jacobians of these quadro-cubic Cremona transformations (10,9) and their zero levels $P_{5,1,2} = 0$ are Calabi-Yau 3-folds in \mathbb{P}^4 .

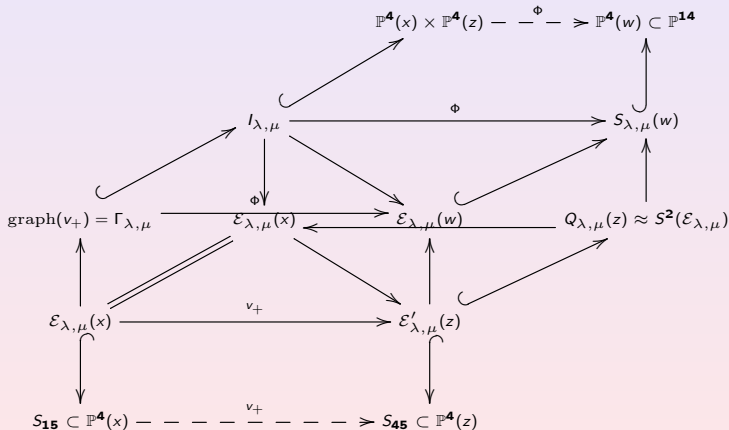
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Cremona transformations in \mathbb{P}^4

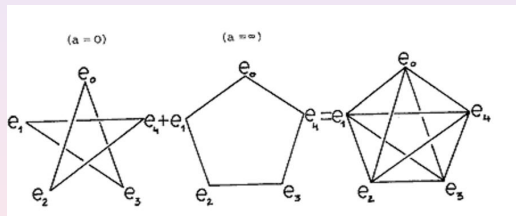


The conditions under which a **general** Cremona transformation (9) on \mathbb{P}^4 gives the Poisson morphism from $q_{5,1}(\mathcal{E})$ to some H -invariant quadratic Poisson algebra read like the following algebraic system:

$$\begin{cases} -a^3\lambda + 4\lambda^4a + 2\lambda^5a^2 + 2\lambda^3 - 2a^2 + a^6\lambda^4 = 0 \\ -1 + 2a^2\lambda^2 - a^3\lambda^3 + 2a\lambda = 0 \end{cases} \quad (11)$$

The system has two classes of solutions: $a\lambda = -1$ and $a = \frac{3 \pm \sqrt{5}}{2\lambda}$ for each λ satisfies to the equation $\lambda^{10} + 11\lambda^5 - 1 = 0$.

These exceptional solutions correspond to the vertexes of the **Klein icosahedron** inside $\mathbb{S}^2 = \mathbb{P}^1$ and the associated singular curves forms **pentagons** (the following figures belong to K. Hulek):



Each pentagon corresponds to a degeneration of the Odesskii-Feigin-Sklyanin algebra $q_{5,2}(\mathcal{E})$ which is (presumably) new examples of H -invariant quadratic Poisson structures on \mathbb{C}^5 .

If $n = 7$ there are three non-isomorphic Sklyanin-Odesskii-Feigin Elliptic Algebras - $Q_{7,1}(\mathcal{E})$, $Q_{7,2}(\mathcal{E})$ and $Q_{7,3}(\mathcal{E})$ with their Poisson counterparts - $q_{7,1}(\mathcal{E})$, $q_{7,2}(\mathcal{E})$ and $q_{7,3}(\mathcal{E})$. The corresponding algebro-geometric objects are:

- three H_7 -invariant elliptic curves \mathcal{E}_a in \mathbb{P}^6 parametrizing by points a of Klein quartic $K : \lambda^3\mu - \mu^3\nu - \nu^3\lambda = 0$;
- if $a = (\lambda, \mu, \nu)$ is a cusp - the curves degenerate to a configuration of lines;
- three H_7 -invariant elliptic ruled surfaces S_a (second symmetric products $S^2(\mathcal{E}_a)$ or the secant of \mathcal{E}_a);
- seven quadrics containing this surfaces define a Cremona transformations (which are birational Poisson morphisms between the quadratic Poisson algebras $q_{7,1}(\mathcal{E})$, $q_{7,2}(\mathcal{E})$ and $q_{7,3}(\mathcal{E})$).

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FIN

THANK YOU FOR YOUR ATTENTION!