Integrable magnetic geodesic flows on 2-torus: new examples via quasi-linear system of PDEs

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Quasi-linear PDEs

Quasi-linear systems of the form

$$A(U)U(x) + B(U)U_y = 0,$$

$$U_t = A(U)U_x, \qquad U = (u_1, \dots, u_n)^T$$

appears in such areas like

- gas-dynamics
- non-linear elasticity
- integrable geodesic flows on 2-torus

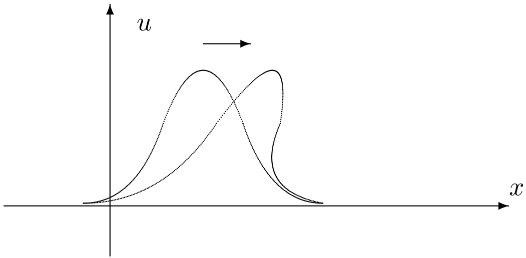
and many others.

Hopf equation

Consider the following equation $u_t + uu_x = 0$. The solution of the Cauchy problem $u|_{t=0} = g(x)$ is given by the implicit formula

$$u(x,t) = g(x - ut).$$

It follows from this formula that the higher any point is placed, the faster it is.



Model of non-linear elasticity

Consider the following equation:

$$u_{tt} + (\sigma(u))_{xx} = 0, \qquad u(t, x+1) = u(t, x). \tag{1}$$

It can be viewed as a compatibility condition of the quasi-linear system of the form:

$$\begin{aligned} u_t &= -v_x, \\ v_t &= (\sigma(u))_x. \end{aligned} \tag{2}$$

Two following cases shall be considered:

Model of non-linear elasticity

In the partial case $\sigma = \frac{u^2}{2}$ the relation with geodesic flow is as follows. Consider the Hamiltonian system

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}.$$

with $H = \frac{1}{2}p^2 + u(t,x)$. Let $F = \frac{1}{3}p^3 + up + v$. Then the condition that F is the first integral is equivalent to the system (2).

Theorem (V.V. Kozlov)
If $u(x,t)$ is doubly–periodic trigonometric polynomial then u is constant.

Model of non-linear elasticity

Theorem (Bialy, M.)
If the function $\sigma(u)$ *is either of quadratic-like or cubic-like type, then any* C^2*-solution* $(u(t,x), v(t,x))$ *of the system (2) defined on the half-cylinder* $[t_0, +\infty) \times \mathbb{S}^1$ *so that*

$$u(t, x+1) = u(t,x), \qquad v(t, x+1) = v(t,x), \qquad t \geq t_0,$$

which has initial values in the Hyperbolic region $U_h = \{u < \alpha\} \cup \{u > \beta\}$ *must be constant.*

Theorem (Bialy, M.)
If the function $\sigma(u)$ *is either of quadratic-like or cubic-like type, then any* C^2*-solution* $(u(t,x), v(t,x))$ *of the system (2) defined on the whole cylinder* $\mathbb{R} \times \mathbb{S}^1$ *so that*

$$u(t, x+1) = u(t,x), \qquad v(t, x+1) = v(t,x)$$

must be constant.

Model of non-linear elasticity

These theorems follow from the following facts. The system (2) can be written in the form:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + A(u,v)\begin{pmatrix} u \\ v \end{pmatrix}_x = 0, \qquad A = \begin{pmatrix} 0 & 1 \\ -\sigma'(u) & 0 \end{pmatrix}.$$

In the hyperbolic region U_h the matrix A has two real distinct eigenvalues:

$$\lambda_1 = \sqrt{-\sigma'(u)}, \qquad \lambda_2 = -\sqrt{-\sigma'(u)}.$$

Riemannian invariants have the form:

$$r_1 = v - \int_u^\alpha \sqrt{-\sigma'(u)}du, \qquad r_2 = v + \int_u^\alpha \sqrt{-\sigma'(u)}du,$$

$$(r_i)_t + \lambda_i (r_i)_x = 0, \qquad i = 1, 2.$$

Model of non-linear elasticity

The crucial fact here is that both eigenvalues are genuinely non-linear in U_h by the formulas:

$$(\lambda_1)_{r_1} = (\lambda_2)_{r_2} = \frac{\sigma^{''}(u)}{4\sigma^{'}(u)} \neq 0.$$

Along characteristics we get the following Riccati equations:

$$L_{v_1}(z_1) + kz_1^2 = 0, \qquad L_{v_2}(z_2) + kz_2^2 = 0,$$

where

$$z_1 = (r_1)_x(-\sigma^{'}(u))^{\frac{1}{4}}, \quad z_2 = (r_2)_x(-\sigma^{'}(u))^{\frac{1}{4}}, \quad k = -\frac{\sigma^{''}}{4(-\sigma^{'}(u))^{\frac{5}{4}}}$$

and

$$L_{v_1} = \partial_t + \lambda_1 \partial_x, \quad L_{v_2} = \partial_t + \lambda_2 \partial_x$$

in what follows non-existence of periodic non-constant solutions.

Integrable geodesic flow on the 2-torus

Let

$$ds^2 = g_{ij}(x)dx^i dx^j, \qquad i,j = 1,2$$

be a Riemannian metric on $\mathbb{T}^2$. The geodesic flow is called *integrable* if the Hamiltonian system

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \qquad H = \frac{1}{2} g^{ij} p_i p_j$$

possesses an additional first integral $F : T^*\mathbb{T}^2 \to \mathbb{R}$ such that

$$\dot{F} = \{F, H\} = \sum_{j=1}^{2} \left(\frac{\partial F}{\partial x^j} \frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial H}{\partial x^j} \right) = 0$$

and F is functionally independent with H almost everywhere.

Polynomial in momenta first integrals

It is known that there exist metrics of two types on the 2-torus with an integrable geodesic flow, namely:

$$ds^2 = \Lambda(\alpha x^1 + \beta x^2)((dx^1)^2 + (dx^2)^2),$$

$$ds^2 = (\Lambda_1(x^1) + \Lambda_2(x^2)((dx^1)^2 + (dx^2)^2).$$

These metrics correspond to existence of an additional polynomial in momenta first integral of the first or of the second degree.

Hypothesis about the degree of first integrals (V.V. Kozlov).
The maximal degree of any *irreducible* polynomial in momenta first integral of geodesic flow on a surface of genus g seems to be not larger than $4 - 2g$.

Integrable geodesic flow on the 2-torus

Theorem (Bialy, M.)
If the Hamiltonian system has an integral F which is a homogeneous polynomial of degree n, then on the covering plane $\mathbb{R}^2$ there exist the global semi-geodesic coordinates (t, x) such that

$$ds^2 = g^2(t,x)dt^2 + dx^2, \qquad H = \frac{1}{2}(\frac{p_1^2}{g^2} + p_2^2)$$

and F can be written in the form:

$$F_n = \sum_{k=0}^{n} \frac{a_k(t,x)}{g^{n-k}} p_1^{n-k} p_2^k.$$

Here the last two coefficients can be normalized by the following way:

$$a_{n-1} = g, \; a_n = 1.$$

Integrable geodesic flow on the 2-torus

The condition $\{F, H\} = 0$ *is equivalent to the quasi-linear PDEs*

$$U_t + A(U)U_x = 0, \tag{1}$$

where $U^T = (a_0, \ldots, a_{n-1}),\ a_{n-1} = g,$

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & a_1 \\ a_{n-1} & 0 & \ldots & 0 & 0 & 2a_2 - na_0 \\ 0 & a_{n-1} & \ldots & 0 & 0 & 3a_3 - (n-1)a_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_{n-1} & 0 & (n-1)a_{n-1} - 3a_{n-3} \\ 0 & 0 & \ldots & 0 & a_{n-1} & na_n - 2a_{n-2} \end{pmatrix}.$$

Semi-Hamiltonian systems

Theorem (Bialy, M.)
(1) is semi-Hamiltonian system. Namely, there is a regular change of variables

$$U \mapsto (G_1(U), \dots, G_n(U))$$

such that for some $F_1(U), \dots, F_n(U)$ *the following conservation laws hold:*

$$(G_i(U))_x + (F_i(U))_y = 0, \qquad i = 1, \dots, n.$$

Moreover, in the hyperbolic domain, where eigenvalues $\lambda_1, \dots, \lambda_n$ *of A(U) are real and pairwise distinct, there exists a change of variables*

$$U \mapsto (r_1(U), \dots, r_n(U))$$

such that the system can be written in Riemannian invariants:

$$(r_i)_x + \lambda_i(r)(r_i)_y = 0, \qquad i = 1, \dots, n.$$

Geodesic flow on the 2-torus in elliptic region

The question of integrability of the geodesic flow is convenient to study in semi-geodesic coordinates:

$$ds^2 = g^2(t,x)dt^2 + dx^2, \qquad H = \frac{1}{2}(\frac{p_1^2}{g^2} + p_2^2)$$

and F can be written in the form:

$$F_n = \sum_{k=0}^{n} \frac{a_k(t,x)}{g^{n-k}} p_1^{n-k} p_2^k.$$

Theorem (Bialy, M.)
Let $n = 4$, then in the elliptic regions the following alternative holds: either metric is flat or F_4 is reducible, that is it can be expressed as:

$$F_4 = k_1 F_2^2 + 2k_2 H F_2 + 4k_3 H^2$$

where F_2 is a polynomial of degree 2 which is an integral of the geodesic flow in the elliptic region and k_i are constants.

Geodesic flow on the 2-torus in elliptic region

The following theorem is crucial in proof of the previous one.

Theorem (Bialy, M.)
Assume that $\Omega_e = \mathbb{T}^2$ *and assume that for all* (t, x) *the polynomial* G_4 *has 4 distinct roots, 2 – complex conjugate* $s_{1,2} = \alpha \pm i\beta$ *and 2 real* $s_{3,4}$*. Assume also that the imaginary part of Riemann invariants* $r_{1,2}$ *does not vanish. Then the real eigenvalues* $\lambda_{3,4} = gs_{3,4}$ *are necessarily genuinely non-linear and therefore the corresponding Riemann invariants are constants. In particular all* a_i *must be constant, and so the metric is flat.*

Magnetic geodesic flow

Consider Hamiltonian system

$$\dot{x}^j = \{x^j, H\}_{mg}, \qquad \dot{p}_j = \{p_j, H\}_{mg}, \qquad j = 1, 2$$

on a 2-torus in magnetic field with $H = \frac{1}{2} g^{ij} p_i p_j$ and the Poisson bracket:

$$\{F, H\}_{mg} = \sum_{i=1}^{2} \left(\frac{\partial F}{\partial x^i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial x^i} \right) + \Omega(x^1, x^2) \left(\frac{\partial F}{\partial p_1} \frac{\partial H}{\partial p_2} - \frac{\partial F}{\partial p_2} \frac{\partial H}{\partial p_1} \right). \tag{6}$$

Example

Let $ds^2 = \Lambda(y)(dx^2 + dy^2)$ and the magnetic form $\omega = -u'(y) dx \wedge dy$. Then the magnetic geodesic flow is integrable and the first integral is linear in momenta:

$$F_1 = p_1 + u(y).$$

Main results

Theorem

There exist real analytic Riemannian metrics on the 2-torus which are arbitrary close to the Liouville metrics (and different from them) and a non-zero analytic magnetic fields such that magnetic geodesic flows on the energy level $\{H = \frac{1}{2}\}$ have polynomial in momenta first integral of degree two.

Theorem

Consider the magnetic flow of the Riemannian metric $ds^2 = \Lambda(dx^2 + dy^2)$ with the non-zero magnetic form ω. Suppose the magnetic flow admits a first integral F_2 on all energy levels such that F_2 is quadratic in momenta. Then in some coordinates the functions Λ, Ω have the form of the Example 1, so there exists another integral F_1 linear in momenta and F_2 can be written as a combination of H and F_1.

Magnetic geodesic flow

Choose the conformal coordinates (x, y), such that $ds^2 = \Lambda(x, y)(dx^2 + dy^2)$, $H = \frac{p_1^2+p_2^2}{2\Lambda}$. On the fixed energy level $H = \frac{1}{2}$ one can parameterize momenta by the following way:

$$p_1 = \sqrt{\Lambda}\cos\varphi, \qquad p_2 = \sqrt{\Lambda}\sin\varphi.$$

Hamiltonian equations take the form

$$\dot{x} = \frac{\cos\varphi}{\sqrt{\Lambda}}, \qquad \dot{y} = \frac{\sin\varphi}{\sqrt{\Lambda}}, \qquad \dot{\varphi} = \frac{\Lambda_y}{2\Lambda\sqrt{\Lambda}}\cos\varphi - \frac{\Lambda_x}{2\Lambda\sqrt{\Lambda}}\sin\varphi - \frac{\Omega}{\Lambda}.$$

We shall search F in the form

$$F(x, y, \varphi) = \sum_{k=-N}^{k=N} a_k(x, y)e^{ik\varphi}. \tag{7}$$

Here $a_k = u_k + iv_k, a_{-k} = \bar{a}_k$. Condition $\dot{F} = \{F, H\}_{mg} = 0$ is equivalent to the following equation

$$F_x\cos\varphi + F_y\sin\varphi + F_\varphi\left(\frac{\Lambda_y}{2\Lambda}\cos\varphi - \frac{\Lambda_x}{2\Lambda}\sin\varphi - \frac{\Omega}{\sqrt{\Lambda}}\right) = 0. \tag{8}$$

Magnetic geodesic flow

Substituted (7) to (8), all the coefficients of $e^{ik\varphi}$ must be equal to zero. One obtains

$$\frac{\Lambda_y}{2\Lambda}\frac{i(k-1)a_{k-1}+i(k+1)a_{k+1}}{2}-\frac{\Lambda_x}{2\Lambda}\frac{i(k-1)a_{k-1}-i(k+1)a_{k+1}}{2i}+$$

$$+\frac{(a_{k-1})_x+(a_{k+1})_x}{2}+\frac{(a_{k-1})_y-(a_{k+1})_y}{2i}-\frac{ik\Omega a_k}{\sqrt{\Lambda}}=0, \quad (9)$$

where $k=0,\ldots,N+1,\ a_k=0$ while $k>N$.

One can eliminate Ω from this system thus obtaining the quasilinear PDEs on a_j and Λ of a kind

$$A(U)U_x+B(U)U_y=0, \quad (10)$$

where $U=(\Lambda,u_0,u_1,\ldots,u_{N-1},v_1,\ldots,v_{N-1})^T$.

Semi-Hamiltonian system

Let $N = 2$. Then (10) takes the form

$$A(U)U_x + B(U)U_y = 0,$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ f & 0 & \Lambda & 0 \\ 2 & 1 & 0 & \frac{g}{2} \\ 0 & 0 & 0 & -\frac{f}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -g & 0 & 0 & -\Lambda \\ 0 & 0 & -\frac{g}{2} & 0 \\ 2 & -1 & \frac{f}{2} & 0 \end{pmatrix}, \qquad (11)$$

$U = (\Lambda, u_0, f, g)^T, \qquad f = \frac{u_1}{\sqrt{\Lambda}}, \qquad g = \frac{v_1}{\sqrt{\Lambda}}.$

Magnetic field takes the form: $\Omega = \frac{1}{4}(g_x - f_y).$

System (10) is proved to be semi-Hamiltonian (by M. Bialy, A.E. Mironov) and Egorov system (by S.V. Agapov, M. Bialy, A.E. Mironov) for any N.

Crucial construction

One can check that

$$U_0(x,y) = \begin{pmatrix} \Lambda_1(x) + \Lambda_2(y) \\ 2\Lambda_2(y) - 2\Lambda_1(x) \\ 0 \\ 0 \end{pmatrix} \qquad (12)$$

is the solution of the system (11), where $\Lambda_1(x)$ and $\Lambda_2(y)$ are periodic positive functions: $\Lambda_1(x+1) = \Lambda_1(x),\ \Lambda_2(y+1) = \Lambda_2(y)$. This solution corresponds to the case of geodesic flow of the Liouville metric with zero magnetic field having the first integral of the second degree of the form

$$F_2 = \frac{\Lambda_2(y)p_1^2 - \Lambda_1(x)p_2^2}{\Lambda_1(x) + \Lambda_2(y)}.$$

Λ_1 and Λ_2 are assumed to be real analytic functions.

Introduce the following evolution equations:

$$U_t = A_1(U)U_x + B_1(U)U_y, \tag{13}$$

where

$$A_1 = \begin{pmatrix} g & 0 & 0 & \Lambda \\ -2g & g & 0 & -2\Lambda \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} f & 0 & \Lambda & 0 \\ 2f & f & 2\Lambda & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This system defines the symmetry of the system (11) so that the flow of (13) transforms solutions to solutions as we shall prove below.
Next we apply the following consequence of Cauchy–Kowalevskaya theorem:

Lemma

The Cauchy problem for the system (13) with the initial data

$$U(x,y,t)\mid_{t=0} = U_0(x,y) \tag{14}$$

has a unique analytic periodic $(U(x+1,y,t) = U(x,y+1,t) = U(x,y,t))$ *solution for* t *small enough.*

Let us prove that $U(x, y, t)$ constructed in Lemma 1 is a solution of our system (9) for all small t. We denote by $\tilde{V}(x, y, t)$ the following real analytic vector function

$$\tilde{V} = A(U)U_x + B(U)U_y.$$

By our construction $\tilde{V}(x, y, 0) = 0$. We have to prove that $\tilde{V} \equiv 0$. Denote

$$\tilde{V} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}, \qquad V = \begin{pmatrix} V_2 \\ V_3 \\ V_4 \end{pmatrix}.$$

By direct calculations using (13) one can check that $\tilde{V}$ satisfies the following system of equations:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}_t = A_2 \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}_x + B_2 \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}_y + C_2 \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} + D_2 \begin{pmatrix} V_1^2 \\ V_1 V_2 \\ V_1 V_3 \\ V_2 V_3 \end{pmatrix}. \qquad (15)$$

Here

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -g\Lambda & g & 0 & -2\Lambda \\ 0 & 0 & 0 & 0 \\ 2\Lambda & -2 & f & g \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ f\Lambda & f & 2\Lambda & 0 \\ 2\Lambda & 2 & f & g \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & -2\Lambda_x \\ c_4 & 0 & f_y & -f_x \\ c_5 & 0 & -g_y & g_x \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{f\Lambda}{g} & -\frac{f}{g} & -\frac{4\Lambda}{g} & -\frac{4}{g} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$c_1 = \frac{2\Lambda f_x f + (f^2 - g^2 + 4\Lambda)\Lambda_x + 2\Lambda u_{0_x}}{g}, \; c_2 = \frac{2(g g_x + 2\Lambda_x + u_{0_x})}{g},$$

$$c_3 = \frac{4\Lambda f_x + 2f\Lambda_x}{g}, \; c_4 = 4\Lambda_y + \frac{1}{2} f(f_y - g_x), \; c_5 = 4\Lambda_x - \frac{1}{2} g(f_y - g_x).$$

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