Explicit formula for Virasoro singular vector

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Virasoro algebra, Verma modules

The Virasoro algebra Vir is an infinite-dimensional Lie algebra, defined by its basis $\{C, L_i,\ i \in \mathbb{Z}\}$ and commutation relations:

$$[L_i, L_j] = (i-j)L_{i+j} + \frac{i^3 - i}{12}\delta_{-i,j}C, \quad [L_i, C] = 0,\ \forall i \in \mathbb{Z}.$$

A module $V(h,c)$ over Vir is called a Verma module, if
1) it is free over the universal enveloping algebra $U(Vir^-)$ of the subalgebra $Vir^-=\langle L_i, i<0\rangle$, and
2) it is generated by some vector v (vacuum vector) such that

$$Cv = cv,\ L_0 v = hv, \quad L_i v = 0,\ i > 0,$$

where $c, h \in \mathbb{C}$.

Positive grading of a Verma module

As a vector space $V(h,c)$ can be defined by its infinite basis

$$v,\ L_{-i_1}\dots L_{-i_s}v, \quad i_1 \geq i_2 \geq \dots \geq i_s,\ s \geq 1.$$

The module $V(h,c)$ is positively graded:

$$V(h,c) = \bigoplus_{n=0}^{+\infty} V_n(h,c),$$

$$V_n(h,c) = \langle L_{-i_1}\dots L_{-i_s}v,\ i_1+\ldots+i_s=n \rangle.$$

$V_n(h,c)$ is the eigen-subspace of the operator L_0, with the eigen-value $(h+n)$:

$$L_0(L_{-i_1}\dots L_{-i_s}v) = (h+i_1+\dots+i_s)L_{-i_1}\dots L_{-i_s}v.$$

Singular vectors and reducibility of Verma modules

A nontrivial vector $w \in V(h,c)$ is called *singular*, if

$$L_i w = 0,\ \forall\ i > 0.$$

$$L_3 w = L_2 L_1 w - L_1 L_2 w,\ L_4 w = 2L_3 L_1 w - 2L_1 L_3 w, \dots$$

Hence a vector $w \in V(h,c)$ is singular iff

$$L_1 w = L_2 w = 0.$$

A homogeneous singular vector $w_n \in V_n(h,c)$ generates in $V(h,c)$ a submodule that is isomorphic to Verma module $V(h+n,c)$.
Moreover the module $V(h,c)$ is reducible if and only if it contains a singular vector not in $\mathbb{C}v$:

General problems

The two principle problems in the Virasoro representation theory are:

- For which parameters h, c the module $V(h, c)$ is reducible?
- What are the singular vectors in a reducible $V(h, c)$?

Level $n = 1$.

One-dimensional $V_1(h,c) = \langle L_{-1}v \rangle$

There is a singular vector $w_1 \in V_1(h,c)$ if and only if $h=0$:

$$L_1 L_{-1} v = 2L_0 v = 2hv, \quad L_2 L_{-1} v = 0.$$

Level $n = 2$.

Two-dimensional $V_2(h,c) = \langle L_{-1}^2 v, L_{-2} v \rangle$.
Elementary calculations:
a) $L_1 L_{-2} v = 3L_{-1} v$;
b) $L_1 L_{-1}^2 v = 2L_0 L_{-1} v + L_{-1} L_1 L_{-1} v = 2(h+1) L_{-1} v + 2h L_{-1} v$;

$$L_1 w_2 = 0 \;\Leftrightarrow\; w_2 = L_{-1}^2 v - \frac{2}{3}(2h+1) L_{-2} v,$$

c) $L_2 L_{-2} v = (4L_0 + \frac{2^3-2}{12} C) v = \left(4h + \frac{c}{2}\right) v$;
d) $L_2 L_{-1}^2 v = 3L_1 L_{-1} v + L_{-1} L_2 L_{-1} v = 6hv$.

$$L_2 w_2 = 0 \;\Leftrightarrow\; 6h - \frac{2}{3}(2h+1)\left(4h + \frac{c}{2}\right) = 0.$$

Level $n = 2$.

The solutions (c, h) of the quadratic equation

$$6h - \frac{2}{3}(2h+1)\left(4h + \frac{c}{2}\right) = 0$$

can be parametrized:

$$c(t) = 13 + 6t + 6t^{-1}, \quad h(t) = -\frac{3}{4}t - \frac{1}{2}, \ \ 0 \neq t \in \mathbb{C}.$$

And we have a very simple formula for the singular vector w_2:

$$w_2 = L_{-1}^2 v - \frac{2}{3}(2h+1)L_{-2}v = L_{-1}^2 v + tL_{-2}v.$$

Levels $n = 3, 4$

$V_3(h,c) = \langle L_{-1}^3 v, L_{-2}L_{-1}v, L_{-3}v \rangle$.
A singular vector w_3 exists iff for some $0 \neq t \in \mathbb{C}$ the parameters are

$$c(t) = 13 + 6t + 6t^{-1}, \quad h(t) = -2t - 1,$$

and the singular vector w_3 is unique up to some scalar constant:

$$w_3 = L_{-1}^3 v + 4tL_{-2}L_{-1}v + (4t^2 + 2t)L_{-3}v.$$

The next level $n = 4$ is much more complicated:

$$V_4(h,c) = \langle L_{-1}^4 v, L_{-2}L_{-1}^2 v, L_{-3}L_{-1}v, L_{-2}^2 v, L_{-4}v \rangle.$$

It follows from the Kac theorem and from the results by Feigin and Fuchs that there is a singular vector w_n in $V_n(h, c)$ iff there exist $p, q \in \mathbb{N}$ and $t \neq 0$ such that

$$pq = n,$$
$$c = c(t) = 13 + 6t + 6t^{-1},$$
$$h = h_{p,q}(t) = \tfrac{1-p^2}{4}t + \tfrac{1-pq}{2} + \tfrac{1-q^2}{4}t^{-1}.$$

For fixed $p, q \in \mathbb{N}$ and a fixed $t \in \mathbb{C}$ the Verma module $V(h_{p,q}(t), c(t))$ containes a singular vector $w_{p,q}(t)$ of degree pq,

$$w_{p,q}(t) = S_{p,q}(t)v = \sum_{i_1+\dots+i_s=pq} P_{p,q}^{i_1,\dots,i_s}(t) L_{-i_1} \dots L_{-i_s} v,$$

The coefficients $P_{p,q}^{i_1,\dots,i_s}(t)$ depend polynomially on t and t^{-1}. We assume $P_{p,q}^{1,\dots,1}(t) \equiv 1$. It is obvious that $S_{p,q}(t) = S_{q,p}(t^{-1})$.

Benoist's and St-Aubin's formula

In 1988 Benoist and St-Aubin found a beatifull explicit expression for the series $S_{1,p}(t)$:

$$S_{1,p}(t)= \sum_{i_1+\ldots+i_s=p} \frac{(p-1)!^2 t^{s-p}}{\prod_{l=1}^{s-1}\left((\sum_{q=1}^{l} i_q)(p-\sum_{q=1}^{l} i_q)\right)} L_{-i_1}\ldots L_{-i_s}, \qquad (1)$$

First examples: $S_{1,1}(t)=L_{-1}$, $S_{1,2}(t)=L_{-1}^2+t^{-1}L_{-2}$ and

$$S_{1,3}(t)=L_{-1}^3+2t^{-1}(L_{-1}L_{-2}+L_{-2}L_{-1})+4t^{-2}L_{-3},$$

$$S_{1,4}(t)=L_{-1}^4+t^{-1}(3L_{-1}^2L_{-2}+4L_{-1}L_{-2}L_{-1}+3L_{-2}L_{-1}^2)+ \\ +t^{-2}(12L_{-1}L_{-3}+9L_{-2}^2+12L_{-3}L_{-1})+36t^{-3}L_{-4}.$$

The key idea by Benoist and St-Aubin was a proposal to seek the operator $S_{1,p}(t)$ in the form of linear combination of all homogeneous monomials $L_{-i_1}L_{-i_2}\dots L_{-i_s}, i_1{+}i_2{+}\dots{+}i_s{=}p$ in $U(Vir^{-})$ and not only of basic ones with $i_1 \geq i_2 \geq \dots \geq i_s \geq 1$.

Later Bauer, Di Francesco, Itzykson and Zuber published an elegant proof of singularity of vectors $S_{1,p}(t)v$.Moreover, using Benoit-St-Aubin formula as an initial step, they proposed an algorithm for finding all singular vectors. But in practice, this algorithm meets with serious computational difficulties and it is still unclear how to get with its help an explicit formula for all $S_{p,q}(t)$. However, the expressions for $S_{2,2}(t)$ and $S_{2,3}(t)$ found by means of it allowed us to guess the general formula for singular vectors of the entire series $S_{2,p}(t)$. The singularity of these vectors can be established by modifying the scheme of the proof Bauer-Di Francesco-Itzykson-Zuber for the series $S_{1,p}(t)$ in a special way.

First examples for $S_{2,p}$

$$S_{2,1}(t) = L_{-1}^2 + tL_{-2},$$

$$S_{2,2}(t) = L_{-1}^4 + 4tL_{-1}L_{-2}L_{-1} + \frac{(1-t^2)}{t}(L_{-1}^2L_{-2} + L_{-2}L_{-1}^2) + \\ + \frac{(1+t)(4t-1)}{t}L_{-1}L_{-3} + \frac{(1-t)(4t+1)}{t}L_{-3}L_{-1} + \\ + \frac{(1-t^2)^2}{t^2}L_{-2}^2 + \frac{3(1-t^2)}{t}L_{-4}.$$

Main theorem, M. 2016

Let $V(h_{2,p}(t), c(t)), t \in \mathbb{C}, t \neq 0$ be a Verma module over Vir with $c(t){=}13{+}6t{+}6t^{-1}$, $h_{2,p}(t){=}-\frac{1}{4}(p{-}1{+}t)(t^{-1}(p{+}1){+}3)$, Consider

$$S_{2,p}(t){=}\sum_{i_1+\ldots+i_s=2p} (2p{-}1)!^2(2t)^{s-2p} f_{2,p}(t;i_1,\ldots,i_s)L_{-i_1}\ldots L_{-i_s},$$

$$f_{2,p}(t;i_1,\ldots,i_s){=}\frac{\prod_{r=1}^{2p-1}(p{-}t{-}r)\prod_{m=1}^{s}\left((2t{-}1)(i_m{-}1){+}2p{-}1{-}2\sum_{n=1}^{m-1} i_n\right)}{\prod_{k=0}^{2p-1}(2p{-}1{-}2k)\prod_{l=1}^{s-1}\left((\sum_{n=1}^{l} i_n)(2p{-}\sum_{n=1}^{l} i_n)(p{-}t{-}\sum_{n=1}^{l} i_n)\right)}$$

The vector $S_{2,p}(t)v$ is singular in $V(h_{2,p}(t), c(t))$.

$$
\begin{aligned}
S_{2,3}(t) = {} & L^6_{-1} + \frac{(4-t^2)}{3t}(L^4_{-1}L_{-2}+L_{-2}L^4_{-1}) + \frac{8(1-t^2)}{3t}(L^3_{-1}L_{-2}L_{-1}+ \\
& + \frac{8(1-t^2)}{3t}L_{-1}L_{-2}L^3_{-1}) + 9tL^2_{-1}L_{-2}L^2_{-1} + \frac{(4-t^2)^2}{9t^2}L_{-2}L^2_{-1}L_{-2}+ \\
& + 3(4-t^2)(L^2_{-1}L^2_{-2}+L^2_{-2}L^2_{-1}) + \frac{64(1-t^2)^2}{9t^2}L_{-1}L^2_{-2}L_{-1}+ \\
& + \frac{8(1-t^2)(4-t^2)}{9t^2}(L_{-1}L_{-2}L_{-1}L_{-2}+L_{-2}L_{-1}L_{-2}L_{-1}) + \frac{(4-t^2)^2}{t}L^3_{-2}+ \\
& + \frac{4(4-t^2)(1-t^2)(9-16t^2)}{9t^4}L^2_{-3} + \frac{6(1+t)(4t-1)}{t}L^2_{-1}L_{-3}L_{-1}+ \\
& \frac{6(1-t)(4t+1)}{t}L_{-1}L_{-3}L^2_{-1} + \frac{2(1+t)(2+t)(3-4t)}{3t^2}L^3_{-1}L_3+ \\
& \frac{2(1-t)(2-t)(3+4t)}{3t^2}L_{-3}L^3_{-1} + \frac{16(1-t^2)(1+t)(2+t)(3-4t)}{9t^3}L_{-1}L_{-2}L_3+ \\
& \frac{16(1-t^2)(1-t)(2-t)(3+4t)}{9t^3}L_{-3}L_{-2}L_{-1} + \frac{2(4-t^2)(2+t)(1+t)(3-4t)}{9t^3}L_{-2}L_{-1}L_{-3}+ \\
& + \frac{2(4-t^2)(2-t)(1-t)(3+4t)}{9t^3}L_{-3}L_{-1}L_{-2}+ \\
& + \frac{2(4-t^2)(1-t)(4t+1)}{t^2}L_{-1}L_{-3}L_{-2} + \frac{2(4-t^2)(1+t)(4t-1)}{t^2}L_{-2}L_{-3}L_{-1}+ \\
& + \frac{6(1+t)(2+t)(3t-1)}{t^2}L^2_{-1}L_{-4} + \frac{48(1-t^2)}{t}L_{-1}L_{-4}L_{-1} + \frac{6(1-t)(2-t)(3t+1)}{t^2}L_{-4}L^2_{-1} \\
& + \frac{(4-t^2)(2+t)(1+t)(3t-1)}{t^3}L_{-2}L_{-4} + \frac{(4-t^2)(2-t)(1-t)(3t+1)}{t^3}L_{-4}L_{-2}+ \\
& + \frac{4(1-t^2)(2+t)(8t-1)}{t^3}L_{-1}L_{-5} + \frac{4(1-t^2)(2-t)(8t+1)}{t^3}L_{-5}L_{-1} + \frac{20(1-t^2)(4-t^2)}{t^3}L_{-6}
\end{aligned}
$$

Proof

Let's define the chain of vectors $v^{(0)}, v^{(1)}, \dots, v^{(2p)}$ using the following recurrence relation:

$$v^{(0)}{=}v, \quad v^{(k)} = \sum_{j=1}^{k} (2t)^{1-j} a_{j,k} L_{-j} v^{(k-j)}, \quad k = 1, \dots, 2p,$$

$$a_{1,k} = 2k{-}2p{-}1,$$
$$a_{j,k} = ((j{-}1)(2t{-}1){+}2k{-}2p{-}1) \prod_{l=k-j+1}^{k-1} l(2p{-}l)(l{-}p{-}t),\ 2 \leq j \leq k.$$

Here are the first three vectors of our chain:

$$v^{(0)}{=}v,\ v^{(1)}{=}(1{-}2p)L_{-1}v^{(0)},$$
$$v^{(2)} = (3{-}2p)L_{-1}v^{(1)} + (2t)^{-1}(2t{+}2{-}2p)(2p{-}1)(1{-}p{-}t)L_{-2}v^{(0)}.$$

Lemma (A)

The operator L_1 acts on the vectors $v^{(k)}$ as follows:

$$L_1 v^{(k)} = -(p+2-k+3t)k(2p-k)(k-p-t)(2t)^{-1}v^{(k-1)}, \quad k = 1, \dots, 2p$$

In particular $L_1 v^{(2p)} = 0$.

We prove the lemma by induction on k.
First step $k=1$. We recall that $4h=-(p-1+t)(t^{-1}(p+1)+3)$ in our Verma module

$$L_1\left((1-2p)L_{-1}v^{(0)}\right) = (1-2p)2hv^{(0)} =$$
$$= -(p+1+3t)(2p-1)(1-p-t)(2t)^{-1}v^{(0)}.$$

Lemma (B)

The operator L_2 acts on vectors $v^{(k)}$, $k=1,\ldots,2p$, as follows

$$L_2 v^{(k)} = -(p+4-k+5t)(2t)^{-2} \prod_{l=k-1}^{k} \left(l(2p-l)(l-p-t)\right) v^{(k-2)}.$$

In particular the operator L_2 annihilates vectors $v^{(1)}$ and $v^{(2p)}$.

First step of the proof:
$L_2 v^{(1)} = (1-2p)L_2 L_{-1} v^{(0)} = (1-2p)(3L_1 v^{(0)} + L_{-1} L_2 v^{(0)}) = 0.$

Now we are going to consider Verma modules over the Virasoro algebra with $c = 0$ (Verma modules over the Witt algebra W).

Theorem (Kac, Feigin and Fuchs)

There is a singular vector w_n in the homogeneous subspace $V_n(0,0)$ of the Verma module $V(0,0)$ then and only then when n is equal to some pentagonal number $n = e_{\pm}(k) = \frac{3k^2 \pm k}{2}$.

It follows that if a Verma module $V(h,0)$ (c=0) has a singular vector $w_{p,q}(t)$ it implies that $t{=}-\frac{3}{2}$ or $t{=}-\frac{2}{3}$. We will fix the value $t = -\frac{3}{2}$ and we will write $S_{p,q}$ instead of $S_{p,q}\left(-\frac{3}{2}\right)$. Let us denote by $V\left(\frac{3k^2 \pm k}{2}\right)$ a submodule in $V(0,0)$ generated by a singular vector with the grading $\frac{3k^2 \pm k}{2}$. The submodule $V\left(\frac{3k^2 \pm k}{2}\right)$ is isomorphic to the Verma module $V\left(\frac{3k^2 \pm k}{2}, 0\right)$.

Lemma (Rocha-Carridi-Wallach, Feigin-Fuchs)

The system of submodules $V\left(\frac{3k^2\pm k}{2}\right)$ *has the following important properties:*

1) the sum $V(1)+V(2)$ *is the subspace of codimension one in* $V(0)$*;*

2) $V\left(\frac{3k^2-k}{2}\right)\cap V\left(\frac{3k^2+k}{2}\right)=$
$V\left(\frac{3(k+1)^2-(k+1)}{2}\right)+V\left(\frac{3(k+1)^2+(k+1)}{2}\right), k\geq 1.$

The vectors $L_{-1}v$ and $\left(L_{-1}^2 - \frac{2}{3}L_{-2}\right)v$ are singular in the module $V(0,0)$ with the gradings 1 and 2 respectively.

Consider a submodule $V(1)$ generated by $w_1 = L_{-1}v$. It is isomorphic to the Verma module $V(1,0)$ and it containes a singular vector $S_{1,4}w_1$.

A Verma module $V(2,0) = V(2)$ (generated by the vector $w_2 = S_{1,2}v$) containes a singular vector $S_{3,1}w_2$.
Vectors $S_{1,4}w_1$ and $S_{3,1}w_2$ are both singular and they are at the level $n=5$ in the Verma module $V(0,0)$. Hence they coincide

$$w_5 = S_{1,4}w_1 = S_{3,1}w_2.$$

Similarly, one can check the other equality

$$w_7 = S_{3,2}w_1 = S_{1,5}w_2.$$

$V(1) + V(2)$ has codimension one in $V(0,0)$: $v \notin V(1) + V(2)$.

We see that singular vectors $w_5, w_7 \in V(1) \cap V(2)$ as well as the sum $V(5) + V(7)$ of submodules generated by w_5 and w_7 respectively:

$$V(5) + V(7) \subset V(1) \cap V(2).$$

The intersection $V(5) \cap V(7)$ containes two singular vectors

$$w_{12} = S_{1,7} w_5 = S_{5,1} w_7, \quad w_{15} = S_{5,2} w_5 = S_{1,8} w_7.$$

The inclusions of submodules $V\left(\frac{3k^2 \pm k}{2}\right)$ provides us with an exact sequence (Rocha-Carridi-Wallach, Feigin-Fuchs):

$$\to V\left(\frac{3(k+1)^2-(k+1)}{2}\right)\oplus V\left(\frac{3(k+1)^2+(k+1)}{2}\right) \xrightarrow{\delta_{k+1}} V\left(\frac{3k^2-k}{2}\right)\oplus V\left(\frac{3k^2+k}{2}\right)\to\dots$$

$$\dots \xrightarrow{\delta_3} V(5)\oplus V(7) \xrightarrow{\delta_2} V(1)\oplus V(2) \xrightarrow{\delta_1} V(0) \xrightarrow{\varepsilon} \mathbb{C}\to 0 \qquad (2)$$

where δ_k are defined with the aid of operators $S_{p,q} \in U(W^-)$:

$$\delta_{k+1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} S_{1,3k+1} & S_{2k+1,2} \\ -S_{2k+1,1} & -S_{1,3k+2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},\ k \geq 1;$$
$$\delta_1\begin{pmatrix} x \\ y \end{pmatrix} = (S_{1,1}, S_{1,2})\begin{pmatrix} x \\ y \end{pmatrix}, \qquad (3)$$

and ε is a projection to the one-dimensional $\mathbb{C}$-submodule generated by the vector v.

Theorem (Rocha-Carridi-Wallach, Feigin-Fuchs)

The exact sequence (2) considered as a sequence of W^--modules is a free resolution of the one-dimensional trivial W^--module $\mathbb{C}$.

Corollary (Feigin-Fuchs)

Let V be a W^--module. Then the cohomology $H^*(W^-, V)$ is isomorphic the cohomology of the following complex:

$$\ldots \xleftarrow{d_{k+1}} V \oplus V \xleftarrow{d_k} V \oplus V \xleftarrow{d_{k-1}} \ldots \xleftarrow{d_1} V \oplus V \xleftarrow{d_0} V, \tag{4}$$

with the differentials

$$\begin{aligned} d_k \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} &= \begin{pmatrix} S_{1,3k+1} & -S_{2k+1,1} \\ S_{2k+1,2} & -S_{1,3k+2} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \; k \geq 1; \\ d_0(m) &= \begin{pmatrix} S_{1,1}m \\ S_{1,2}m \end{pmatrix}, \;\; m, m_1, m_2 \in V. \end{aligned} \tag{5}$$