# From Quantum Groups to Noncommutative Geometry

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# Plan

1. Calculus on $U(gl(m))$
2. Quantum Groups and Braided Geometry
3. Calculus on the algebra $U(u(2)_\hbar)$
4. Noncommutative quantization of models

Consider the algebra $U(gl(m))$. In order to represent it as result of a quantization of the commutative algebra $Sym(gl(m))$ we introduce a quantization parameter $\hbar$ in the front of the Lie bracket. The corresponding algebra is denoted $U(gl(m)_\hbar)$.

**Example** Let $a, b, c, d$ be the standard generators of the algebra $gl(2)$. Then the multiplication table in the Lie algebra $gl(2)_\hbar$ is

$$[a, b] = \hbar\, b,\ [a, c] = -\hbar\, c,\ [a, d] = 0, \ldots.., [d, c] = \hbar\, c.$$

By using the PBW theorem we can choose similar bases in the algebras $Sym(gl(m))$ and $U(gl(m)_\hbar)$, say, ordered polynomials

$$x_1^{k_1} \dots x_n^{k_n}.$$

Consider a morphism of linear spaces

$$\alpha : Sym(gl(m)) \to U(gl(m)_\hbar)$$

taking any basic element of $Sym(gl(m))$ in a similar element of $U(gl(m)_\hbar)$.

Then introduce a new product in the algebra $Sym(gl(m))$, coming from the algebra $U(gl(m)_\hbar)$

$$f \star_\hbar g = \alpha^{-1}(\alpha(f) \cdot \alpha(g))$$

.

This product can be presented as

$$f \star_\hbar g = f\, g + \hbar c_1(f,g) + \hbar^2 c_2(f,g) + \ldots$$

Then the operator

$$f, g \to \{f, g\} = c_1(f,g) - c_1(g,f)$$

is a Poisson bracket.

Below, we define analogs of partial derivatives on the algebra $U(gl(m)_\hbar)$ in such a way that for $\hbar = 0$ we recover the usual partial derivatives in generators of $Sym(gl(m))$.

Also, we define a quantum analog of the differential algebra $\Omega(Sym(gl(m))$ and this of the de Rham operator. All objects are deformations of their classical counterpart.

What is commonly used approach to constructing a diff calculus on a NC algebra $A$? The differential algebra $\Omega(A)$ on $A$ is usually defined via the de Rham operator satisfying the classical Leibniz rule $d(a\,b) = da\,b + a\,db$ but without the relation $a(db) = (db)a$, which plays the central role in the classical case.

This approach leads to the universal differential algebra which is much bigger than the classical one is, if $A$ is commutative. In our construction we retrieve the classical differential algebra as $\hbar \to 0$.

Fix in the Lie algebra $gl(m)_\hbar$ the standard basis $\{n_i^j\},\ 1 \le i,j \le m$ and the usual Lie bracket

$$[n_i^j, n_k^l] = \hbar(n_i^l \delta_k^j - n_k^j \delta_i^l), 1 \le i,j,k,l, \le m.$$

Now, define "partial derivatives" on the algebra $U(gl(m)_\hbar)$ .

Observe that in the algebra $Sym(gl(m))$ the partial derivatives $\partial_k^l = \partial_{n_l^k}$ are defined via the action on the generators $\partial_k^l(n_i^j) = \delta_i^l \, \delta_k^j$ (i.e. the partial derivatives span the space dual to that $span(n_i^j)$) and the coproduct

$$\Delta(\partial_k^l) = \partial_k^l \otimes 1 + 1 \otimes \partial_k^l.$$

This coproduct means that if we apply a derivative to a product $a\,b \in Sym(gl(m))$ we have

$$\partial_k^l(ab) = \partial_k^l(a)b + a\partial_k^l(b).$$

This property is called Leibniz rule.

By passing to the algebra $U(gl(m)_\hbar)$ we do not change the first property (the pairing) and define the new Leibniz rule by means of the following coproduct

$$\Delta(\partial_i^j) = \partial_i^j \otimes 1 + 1 \otimes \partial_i^j + \hbar \sum_k \partial_k^j \otimes \partial_i^k.$$

So, we have

$$\partial_i^j(ab) = \partial_i^j(a)b + a\partial_i^j(b) + \hbar \sum_k \partial_k^j(a)\partial_i^k(b).$$

**Observe that the partial derivatives commute with each other.**

Denote $\mathcal{D}$ the commutative algebra generated by the partial derivatives $\partial_i^j$.

There exists another way of introducing partial derivatives in the commutative case: via the so-called *permutation relations*.
For instance, consider a commutative polynomial algebra generated by $x_i$. Then the partial derivatives in $\partial^i = \partial_{x_i}$ can be introduced via the relations

$$\partial^j x_i = x_i \partial^j + \delta_i^j.$$

The algebra generated by $x_i$ and $\partial^j$ is called the Weyl-Heisenberg algebra. The simplest example of of such an algebra is $pq - qp = 1$.

These permutation relations can be introduced via the coproduct $\Delta(\partial^j) = \partial^j \otimes 1 + 1 \otimes \partial^j$. Indeed, $\partial^j(a\, b) = \partial^j(a) \otimes b + a \otimes \partial^j(b)$. By canceling $b$ and by taking $a = x_i$, we get $\partial^j x_i = \delta_i^j + x_i \partial^j$.

By applying this method to our NC setting we get the following permutation relations:

$$\partial_i^j \, n_k^l = n_k^l \, \partial_i^j + \delta_i^l \delta_k^j + \hbar(\partial_i^l \delta_k^j - \partial_k^j \delta_i^l).$$

Consider the product $U(gl(m)_\hbar) \otimes \mathcal{D}$ equipped with these permutation relations. It can be equipped with a structure of an associative algebra. In order to multiply $(a \otimes \alpha)$ and $(b \otimes \beta)$ we permute $\alpha$ and $b$ and multiply elements in each algebra. The final algebra is called *Weyl-Heisenberg algebra* and is denoted $\mathcal{W}(U(gl(m)_\hbar)$.

Sometimes, it is possible to define permutation relations between two associative algebras $A$ and $B$ without any coproduct in $A$. For instant, in such a way a differential calculus related to QG was constructed.

Once the partial derivatives on an algebra $U(gl(m)_\hbar)$ are defined, we can construct a de Rham complex on this algebra. Introduce "pure differentials" $dn_i^j$ by assuming that they anti-commute with each other.

Let

$$\omega = dn_{i_1}^{j_1} \bigwedge dn_{i_2}^{j_2} \bigwedge \dots \bigwedge dn_{i_k}^{j_k} \otimes f, \;\; f \in U(gl(m)_\hbar)$$

be a $k$-differential. Then by definition we put

$$d\,\omega = dn_{i_1}^{j_1} \bigwedge dn_{i_2}^{j_2} \bigwedge \dots \bigwedge dn_{i_k}^{j_k} \bigwedge \sum_{i,j} dn_i^j \otimes \partial_j^i(f).$$

**Theorem**

$d^2 = 0$

**Proof** It is so since the pure differentials anticommute and the partial derivatives commute.

Let us emphasize that the above calculus is covariant w.r.t. the usual group $GL(m)$.
It is possible to deform the objects of our calculus and to get similar calculus covariant w.r.t. action of the QG.
Let us consider the simplest example of the QG, namely that $U_q(sl(2))$.

This QG is generated by 3 generators $H$, $X$, $Y$ subject to the system

$$[X, Y] = \frac{q^H - q^{-H}}{q - q^{-1}}, \ [H, X] = 2X, \ [H, Y] = -2Y.$$

Being equipped with a coproduct $\Delta : U_q(sl(2)) \to U_q(sl(2))^{\otimes 2}$ introduced by

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \ \Delta(X) = X \otimes q^{\frac{H}{2}} + q^{\frac{-H}{2}} \otimes X,$$

$$\Delta(Y) = Y \otimes q^{\frac{H}{2}} + q^{\frac{-H}{2}} \otimes Y$$

and an antipode $S(H) = -H, \ S(X) = -qX, \ S(Y) = -q^{-1}Y$, it becomes a Hopf algebra.
Below, $q$ is generic.

Also, there exists an element $\mathcal{R} \in U_q(\widehat{sl(2))}^{\otimes 2}$ called universal quantum R-matrix which have some interesting properties. In particular, the operator

$$R = P(\rho_V \otimes \rho_V)\mathcal{R} : V^{\otimes 2} \to V^{\otimes 2},$$

where $P$ is the usual flip and $\rho_V : U_q(sl(2)) \to End(V)$ is the representation of the QG in the fundamental space $V$, is subject to the so-called *braid relation*

$$R_{12}\, R_{23}\, R_{12} = R_{23}\, R_{12}\, R_{23}, \quad R_{12} = R \otimes I, \;\; R_{23} = I \otimes R.$$

We call such operators braidings.

Besides, the operator $R: V^{\otimes 2} \to V^{\otimes 2}$ is subject to the so-called Hecke condition

$$(q\, I - R)(q^{-1}\, I + R) = 0, \;\; q \in \mathbb{K}.$$

So, it has two eigenvalues: $q$ and $-q^{-1}$.
Braidings satisfying the Hecke condition are called Hecke symmetries provided $q \neq \pm 1$ and involutive unless.

Braidings coming from the QG $U_q(gl(m))$ are always Hecke symmetries.

A braiding coming from a QG belonging to one of other series $(B_n, C_n, D_n)$ satisfies a third degree equation and is called Birman-Wenzl-Murakami symmetry.

However, there exist Hecke symmetries and BWM symmetries which are not related to any QG.
Large families of such Hecke symmetries were constructed by myself in the 80's.

In fact, a Hecke symmetry is a special representation of of the Hecke algebra. Remind that the Hecke algebra $H_m(q)$ is generated by a set of generators $\sigma_i$, $i = 1, ..., m-1$ subject to

$$(qId - \sigma_i)(q^{-1}Id + \sigma_i) = 0 \ \forall i,$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \ \ \sigma_i\sigma_j = \sigma_j\sigma_i, \ \ |i-j| \geq 2.$$

So, the mentioned representation is defined by $\rho_R(\sigma_i) = R_{i,i+1}$.

The simplest examples are as follows

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}.$$

They are deformations of the usual flip and the super-flip respectively. The first example comes from the QG $U_q(sl(2))$. However, there is a lot of Hecke symmetries which are deformations neither of flips nor of super-flips.

Given a Hecke symmetry $R$, consider "R-symmetric" and "R-skew-symmetric" algebras

$$Sym_R(V) = T(V)/\langle Im(qI - R)\rangle, \ \bigwedge_R(V) = T(V)/\langle Im(q^{-1}I + R)\rangle$$

and the corresponding Poincaré-Hilbert series

$$P_+(t) = \sum_k dim\, Sym_R^{(k)}(V)t^k, \ P_-(t) = \sum_k dim \bigwedge_R^{(k)}(V)t^k,$$

where the upper index $(k)$ labels homogenous components of these quadratic algebras.

For a generic $q$ the following holds $P_-(-t)P_+(t) = 1$.

**Proposition. (Phung Ho Hai)**

*The HP series $P_-(t)$ (and hence $P_+(t)$) is a rational function:*

$$P_-(t) = \frac{N(t)}{D(t)} = \frac{1 + a_1\,t + \dots + a_r\,t^r}{1 - b_1\,t + \dots + (-1)^s\,b_s\,t^s} = \frac{\prod_{i=1}^r (1 + x_i t)}{\prod_{j=1}^s (1 - y_j t)},$$

*where $a_i$ and $b_i$ are positive integers, the polynomials $N(t)$ and $D(t)$ are coprime, and all the numbers $x_i$ and $y_i$ are real positive.*

We call the couple $(r|s)$ bi-rank. In this sense all Hecke symmetries are similar to super-flips.

Examples. If $R$ comes from the QG $U_q(sl(m))$, then

$$P_-(t) = (1+t)^m.$$

If $R$ is a deformation of the super-flip $P_{m|n}$, then

$$P_-(t) = \frac{(1+t)^m}{(1-t)^n}.$$

Also, there exist "exotic" examples: for any $m \geq 2$ there exits a Hecke symmetry such that

$$P_-(t) = 1 + mt + t^2.$$

Together with R-symmetric and R-skew-symmetric algebras consider two quantum matrix (QM) algebras.

The first algebra is the so-called RTT algebra. This algebra is generated by generators $t_i^j$, $1 \le i, j \le n$ which are organised in a matrix $T = (t_i^j)$, subject to the relation

$$R\, T_1\, T_2 = T_1\, T_2\, R, \quad T_1 = T \otimes I, \quad T_2 = I \otimes T.$$

**Example** Let $R$ be standard Hecke symmetry above (coming from $U_q(sl(2))$). By denoting $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we get the system

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} =$$

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

In this algebra an analog of the determinant $det_q T$ can be defined. If we impose the relation $det_q T = 1$, the quotient becomes a Hopf algebra, i.e. it has a coproduct, a counit, and an antipode.

This Hopf algebra is (restricted) dual to that $U_q(gl(m))$.

There exists another QM algebra of great interest. It is so-called Reflection Equation (RE) algebra defined by

$$R\,L_1\,R\,L_1 - L_1\,R\,L_1\,R = 0,\;\; L_1 = L \otimes I,\;\; L = (l_i^j),\;\; L = (l_i^j).$$

If $R$ comes from the QG $U_q(sl(m))$, the both algebras (RTT one and RE one) are deformations of $Sym(gl(m))$. This means that for a generic $q$ dimensions of the homogeneous components of these quadratic graded algebras are stable and equal to the classical ones.

**Example** related to $U_q(sl(2))$. We put $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \cdot$$

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \cdot$$

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

In it an analog of the determinant $det_q L$ is also well defined.

If we impose the condition $det_q L = 1$, we get a braided Hopf algebra (Sh.Majid). This means that

$$\Delta(ab) = \Delta(a)\,\Delta(b) = (a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 \tilde{b_1} \otimes \tilde{a_2} b_2,$$

where $\tilde{b_1} \otimes \tilde{a_2} = Q(a_2 \otimes b_1)$ and $Q$ is a braiding (similarly to a super-algebra).
However, by contrast with the RTT algebra in the RE algebra the center is "big": it is similar to that of $U(gl(m))$. In particular, it contains all elements $Tr_R L^k$, where $Tr_R$ is the trace associated with $R$. In our simplest example, it equals $Tr_R L = q^{-3} a + q^{-1} d$.

Moreover, the RE algebra, being quadratic, admits a quadratic-linear deformation. We call it *modified RE algebra*.
**Example** $m = 2$. The defining relations of the mRE algebra are

$$qab - q^{-1}ba = \hbar b, \quad q(bc - cb) - \lambda a(d - a) = \hbar(d - a),$$

$$qca - q^{-1}ac = \hbar c, \quad q(cd - cd) - \lambda ca = -\hbar c,$$

$$ad - da = 0, \quad q(db - bd) - \lambda ab = \hbar b$$

where $\lambda = q - q^{-1}$ and $\hbar$ is another parameter of deformation.

The element $l = Tr_R L = q^{-3}a + q^{-1}d$ is central in this algebra as well. It is also valid for $Tr_R L^k$.

Pass to the basis $\{b, c, g = a - d, l\}$. Then the relations become

$$q^2 gb - bg = \hbar(q + q^{-1})b,$$

$$gc - q^2 cg = -\hbar(q + q^{-1})c,$$

$$(q^2 + 1)(bc - cb) + (q^2 - 1)g^2 = \hbar(q + q^{-1})g.$$

The following quadratic element

$$C = (q + q^{-1})^{-1} g^2 + q^{-1} bc + qcb$$

is central. We call it "braided Casimir". It is obtained from $Tr_R L^2$ by canceling $l$.

It is a two parameter deformation of the algebra $Sym(sl(2))$.

By putting $C = c$ we get a deformation of hyperboloid.
If $m > 2$ it is possible to construct similar "braided varieties" by putting

$$Tr_R L = c_1, \;\; Tr_R L^2 = c_2, ..., Tr_R L^m = c_m.$$

They are analogs of orbits in $gl(m)^*$.
Their geometry was described by J.Donin in the case related to the QG and by myself and P.Saponov in a more general setting.

Let us mention the differential calculus on a matrix pseudogroup initiated by Woronowicz. He replaced the algebra $\mathbb{K}[GL(m)]$ by RTT algebra and usual vector fields acting on $\mathbb{K}[GL(m)]$ by the RE algebra and tried to find them permutation relations between them. This approach was developed by L.Faddeev, A.Isaev, P.Pyatov. Their calculus is covariant w.r.t. to the QG.

We modified this construction upon replacing the RTT algebra by the mRE algebra. By passing to the limit $q \to 1$ we arrived to our calculus on the algebra $U(gl(m)_\hbar)$.

Go back to the algebra $U(gl(2)_\hbar)$. More precisely, we deal with its compact form $U(u(2)_\hbar)$ with the following genertors

$$t = \frac{1}{2}(a+d),\ x = \frac{i}{2}(b+c),\ y = \frac{1}{2}(c-b),\ z = \frac{i}{2}(a-d).$$

The multiplications table in the Lie algebra $u(2)_\hbar$ is

$$[x,\, y] = \hbar\, z,\ [y,\, z] = \hbar\, x,\ [z,\, x] = \hbar\, y,\ t \text{ is central.}$$

The algebra $U(u(2)_\hbar)$ play the role of the deformed Minkowski space algebra, where the generator $t$ plays the role of the time, $x, y, z$ play the role of spacial variables.

On this algebra the coproduct mentioned above becomes

$$\Delta(\partial_t) = \partial_t \otimes 1 + 1 \otimes \partial_t + \frac{\hbar}{2}(\partial_t \otimes \partial_t - \partial_x \otimes \partial_x - \partial_y \otimes \partial_y - \partial_z \otimes \partial_z),$$

$$\Delta(\partial_x) = \partial_x \otimes 1 + 1 \otimes \partial_x + \frac{\hbar}{2}(\partial_t \otimes \partial_x + \partial_x \otimes \partial_t + \partial_y \otimes \partial_z - \partial_z \otimes \partial_y),$$

$$\Delta(\partial_y) = \partial_y \otimes 1 + 1 \otimes \partial_y + \frac{\hbar}{2}(\partial_t \otimes \partial_y + \partial_y \otimes \partial_t + \partial_z \otimes \partial_x - \partial_x \otimes \partial_z),$$

$$\Delta(\partial_z) = \partial_z \otimes 1 + 1 \otimes \partial_z + \frac{\hbar}{2}(\partial_t \otimes \partial_z + \partial_z \otimes \partial_t + \partial_x \otimes \partial_y - \partial_y \otimes \partial_x).$$

The corresponding permutation relations are

$$[\partial_t, t] = \frac{\hbar}{2}\partial_t + 1,\ [\partial_t, x] = -\frac{\hbar}{2}\partial_x,\ [\partial_t, y] = -\frac{\hbar}{2}\partial_y,\ [\partial_t, z] = -\frac{\hbar}{2}\partial_z,$$

$$[\partial_x, t] = \frac{\hbar}{2}\partial_x,\ [\partial_x, x] = \frac{\hbar}{2}\partial_t + 1,\ [\partial_x, y] = \frac{\hbar}{2}\partial_z,\ [\partial_x, z] = -\frac{\hbar}{2}\partial_y,$$

$$[\partial_y, t] = \frac{\hbar}{2}\partial_y,\ [\partial_y, x] = -\frac{\hbar}{2}\partial_z,\ [\partial_y, y] = \frac{\hbar}{2}\partial_t + 1,\ [\partial_y, z] = \frac{\hbar}{2}\partial_x,$$

$$[\partial_z, t] = \frac{\hbar}{2}\partial_z,\ [\partial_z, x] = \frac{\hbar}{2}\partial_y,\ [\partial_z, y] = -\frac{\hbar}{2}\partial_x,\ [\partial_z, z] = \frac{\hbar}{2}\partial_t + 1.$$

Thus, we have the Weyl algebra

$$\mathcal{W}(U(u(2)_\hbar)) = U(u(2)_\hbar) \otimes \mathcal{D},$$

equipped with the permutation relations.

By using the coproduct above it is possible to compute the action of any partial derivative on any element of $U(u(2)_\hbar)$.

For instance, $\partial_x(yz) = \frac{\hbar}{2}$. This result turns into the classical one as $\hbar = 0$.

Note that sometimes it is more convenient to use the shifted derivative $\tilde{\partial}_t = \partial_t + \frac{2}{\hbar}$ in the time $t$. Thus, the above coproduct becomes

$$\Delta(\tilde{\partial}_t) = \frac{\hbar}{2}(\tilde{\partial}_t \otimes \tilde{\partial}_t - \partial_x \otimes \partial_x - \partial_y \otimes \partial_y - \partial_z \otimes \partial_z),$$

$$\Delta(\partial_x) = \frac{\hbar}{2}(\tilde{\partial}_t \otimes \partial_x + \partial_x \otimes \tilde{\partial}_t + \partial_y \otimes \partial_z - \partial_z \otimes \partial_y),$$

$$\Delta(\partial_y) = \frac{\hbar}{2}(\tilde{\partial}_t \otimes \partial_y + \partial_y \otimes \tilde{\partial}_t + \partial_z \otimes \partial_x - \partial_x \otimes \partial_z),$$

$$\Delta(\partial_z) = \frac{\hbar}{2}(\tilde{\partial}_t \otimes \partial_z + \partial_z \otimes \tilde{\partial}_t + \partial_x \otimes \partial_y - \partial_y \otimes \partial_x).$$

Now, we want to extend action of the quantum partial derivatives on fractions $a^{-1}\,b$ where $a, b \in U(u(2)_\hbar)$, $a \neq 0$. However, it is not clear how to extend this action on elements $a^{-1}$.
In the classical case if $\partial$ is a derivative and consequently, it is subject to the classical Leibniz rule, we have

$$0 = \partial(1) = \partial(aa^{-1}) = \partial(a)a^{-1} + a\partial(a^{-1}).$$

Thus, we get $\partial(a^{-1}) = -a^{-1}\,\partial(a)a^{-1} = -a^{-2}\,\partial(a)$.
In our present setting we are dealing with another coproduct.

As an example we compute the permutation relations and the element $x^{-1}$ by using this new coproduct.
First, present the permutation relations between the partial derivatives and the element $x$ under the following form

$$\begin{pmatrix} \tilde\partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} x = \begin{pmatrix} x & -\frac{\hbar}{2} & 0 & 0 \\ \frac{\hbar}{2} & x & 0 & 0 \\ 0 & 0 & x & -\frac{\hbar}{2} \\ 0 & 0 & \frac{\hbar}{2} & x \end{pmatrix} \begin{pmatrix} \tilde\partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

In order to compute the permutation relations of the quantum partial derivatives and $x^{-1}$, we have to invert the matrix entering this relation. It is not difficult to do since all entries commute with each other.

In general, we are not able to invert all such matrices. We only succeeded to extend the action of all partial derivatives onto some fractions $a^{-1}b$.

Besides, we introduced the so-called *quantum radius*

$$r_\hbar = \sqrt{x^2 + y^2 + z^2 + h^2}, \;\; \hbar = 2ih.$$

We also succeeded in extending the partial derivatives on any rational function $f(r_\hbar)$ or even a formal series.

Thus, we have

$$\tilde{\partial}_t(r_\hbar^p) = \frac{-i}{2\hbar r_\hbar}((r_\hbar + \hbar)^{p+1} + (r_\hbar - \hbar)^{p+1}) \qquad (1)$$

$$\partial_x(r_\hbar^p) = \frac{x}{r_\hbar}\partial_{r_\hbar}(r_\hbar^p), \;\; \partial_y(r_\hbar^p) = \frac{y}{r_\hbar}\partial_{r_\hbar}(r_\hbar^p), \;\; \partial_z(r_\hbar^p) = \frac{z}{r_\hbar}\partial_{r_\hbar}(r_\hbar^p), \qquad (2)$$

where

$$\partial_{r_\hbar}(f(r_\hbar)) = \frac{f(r_\hbar + \hbar) - f(r_\hbar - \hbar)}{2\hbar}$$

is the *derivative* in the quantum radius.
For instance, if $p = -1$ we have

$$\tilde{\partial}_t(r_\hbar^{-1}) = \frac{-i}{\hbar r_\hbar}, \;\; \partial_{r_\hbar}(r_\hbar^{-1}) = \frac{1}{\hbar^2 - r_\hbar^2}, \;\; \partial_x(r_\hbar^{-1}) = \frac{x}{r_\hbar(\hbar^2 - r_\hbar^2)}.$$

This calculus can be used for a new quantization of dynamical model.
For instance, a "quantum version" of the Klein-Gordon operator is defined in the classical way

$$(\Box - m^2)\, f, \;\; \Box = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$$

Here $f$ is an element of the algebra $U(u(2)_\hbar)$ or its extension and $m$ is the "mass of a NC particle". In a similar way we define NC analogs of other wave (Dirac, Maxwell...) operators with constant coefficients.
By assuming that in the Klein-Gordon equation $(\Box - m^2)\, f = 0$ the "wave function" $f(t, x)$ does not depend on $y$ and $z$, we get a modification of the dispersion relation

$$E = \pm\sqrt{p^2 - m^2} + \frac{\hbar p^2}{1 + \sqrt{1 + (\hbar p)^2}}.$$

For the Schrodinger type equation

$$\left(a\partial_t + b(\partial_x^2 + \partial_y^2 + \partial_z^2) + \frac{q}{r_\hbar}\right) f(t, r_\hbar) = 0$$

we have computed the first correction of the ground state energy.

Let us consider in more detail the Maxwell system and Dirac magnetic monopole.

The Maxwell system consists of 4 equations. The first couple of these equations is (we put $c = 1$)

$$\mathrm{div}\ \mathcal{H} = 0,\ \ \mathrm{curl}\ \mathcal{E} + \partial_t \mathcal{H} = 0,$$

where $\mathcal{E} = (E_1, E_2, E_3)$ and $\mathcal{H} = (H_1, H_2, H_3)$ are vectors of electric and magnetic fields respectively. Also, $\mathrm{div}$ and $\mathrm{curl}$ stand for the divergence and curl respectively.

The second couple of the Maxwell system in vacuum is

$$\mathrm{div}\,\mathcal{E} = 0,\ \ \mathrm{curl}\,\mathcal{H} + \partial_t \mathcal{E} = 0.$$

Let us consider a particular case of this system, giving rise to the Dirac monopole, i.e. we assume $\mathcal{E}$ to be trivial, and consequently $\mathcal{H}$ is assumed to be stationary. Then, we get the following system for the magnetic field

$$\operatorname{div}\mathcal{H} = 0, \ \operatorname{curl}\mathcal{H} = (0,0,0).$$

Also, we assume $\mathcal{H}$ to be spherically symmetric and look for a solution of this system under the form $\mathcal{H} = f(r)(x, y, z)$, where $f(r)$ is a function in the radius $r$. We have $f(r) = g\, r^{-3}$ where $g$ is a constant which is assumed to be real.

Nevertheless, the field $\mathcal{H} = \frac{g}{r^3}(x, y, z)$ is a solution of the equation $\mathrm{div}\,\mathcal{H} = 0$ only on the set $\mathbb{R}^3 \setminus (0, 0, 0)$, whereas, on the whole space $\mathbb{R}^3$ this field meets the equation

$$\mathrm{div}\,\mathcal{H} = 4\,g\pi\delta(r),$$

where $\delta(r)$ is the delta-function on the space $\mathbb{R}^3$ located at the point $(0, 0, 0)$.
The second equation of the above system is also met.

The field $\mathcal{H} = \frac{g}{r^3}(x, y, z)$ is called *Dirac monopole*.

Now, let us apply our NC quantization to this model. We find the following field

$$\mathcal{H} = \frac{g}{r_\hbar(r_\hbar^2 - \hbar^2)}(x, y, z).$$

We call this solution *NC Dirac monopole*. Emphasize that for $\hbar \to 0$ we retrieve the classical Dirac monopole.

In a sense the classical relation

$$\mathrm{div}\mathcal{H} = 4\pi\delta(0)$$

is also valid.

# Many Thanks