# SDYM equations on the self-dual background

L.V. Bogdanov

L.D. Landau ITP RAS

http://www.maths.dur.ac.uk/lms/105/talks/0993bogd.pdf

# Integrable background geometries

**David M.J. Calderbank, SIGMA 10 (2014), 034, 51 pages**

*SDYM equations (and their reductions) are integrable in some nonflat geometries described by dispersionless integrable equations.*

**Atiyah M.F., Hitchin N.J., Singer I.M., Self-duality in four-dimensional Riemannian geometry, Proc. Roy. Soc. London Ser. A 362 (1978), 425–461.**

*There is a curved twistor space as long as the conformal structure on 4-manifold is selfdual. SDYM equations for selfdual conformal structure are integrable by twistor approach.*

We will go opposite direction, starting from dispersionless integrable equations and extending integrable structures (Lax pairs, dressing scheme, the hierarchy) for SDYM equations on geometric background.

## Theorem (Dunajski, Ferapontov and Kruglikov (2014))

*There exist local coordinates* $(z, w, x, y)$ *such that any ASD conformal structure in signature* $(2, 2)$ *is locally represented by a metric*

$$\frac{1}{2}g = dwdx - dzdy - F_y dw^2 - (F_x - G_y)dwdz + G_x dz^2,$$

*where the functions* $F, \; G : M^4 \to \mathbb{R}$ *satisfy a coupled system of third-order PDEs,*

$$\begin{aligned} &\partial_x(Q(F)) + \partial_y(Q(G)) = 0, \\ &(\partial_w + F_y\partial_x + G_y\partial_y)Q(G) + (\partial_z + F_x\partial_x + G_x\partial_y)Q(F) = 0, \end{aligned} \quad (1)$$

*where*

$$Q = \partial_w\partial_x - \partial_z\partial_y + F_y{\partial_x}^2 - G_x{\partial_y}^2 - (F_x - G_y)\partial_x\partial_y.$$

System (1) arises as $[X_1, X_2] = 0$ from the dispersionless Lax pair

$$X_1 = \partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_1\partial_\lambda,$$
$$X_2 = \partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_2\partial_\lambda.$$

Due to compatibility conditions, $f_1$ and $f_2$ can be expressed through $F$ and $G$,

$$f_1 = -Q(G), \quad f_2 = Q(F),$$
$$Q = \partial_w\partial_x - \partial_z\partial_y + F_y{\partial_x}^2 - G_x{\partial_y}^2 - (F_x - G_y)\partial_x\partial_y.$$

Correspondence between ASD conformal structures and integrable system defined by generic commuting vector fields.
Real case with the signature (2,2) or, generally, complex analytic case may be considered.
Reductions:
Dunajski system - null Kähler case, divergence free vector fields
$f_1, f_2 = 0$ (no $\partial_\lambda$ in the vector fields), divergence free - *Plebanski's second heavenly equation* (ASD, Ricci flat)

# Integrability properties of this Lax pair

The hierarchy, Lax-Sato equations, the dressing scheme - Bogdanov, Dryuma and Manakov (2007)
The structure of the hierarchy in terms of vector fields

$$
\begin{aligned}
X_1^n &= \partial_{z^n} - \lambda^n \partial_x + F_1^n(\lambda)\partial_x + G_1^n(\lambda)\partial_y + f_1^n(\lambda)\partial_\lambda, \\
X_2^n &= \partial_{w^n} - \lambda^n \partial_y + F_2^n(\lambda)\partial_x + G_2^n(\lambda)\partial_y + f_2^n(\lambda)\partial_\lambda,
\end{aligned}
$$

where we have two infinite sets of times $z^n$, $w^n$ and two 'basic' variables $x$, $y$, the coefficients of vector fields are polynomials in $\lambda$ of the order $n-1$. Multidimensional version contains $N$ infinite sets of times and $N$ 'basic' variables.

# Extension of the Lax pair

Consider a gauge field **A** in some (matrix) Lie algebra and 'covariant vector fields' $X_1$, $X_2$

$$\nabla_{X_1} = \partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_1\partial_\lambda + A_1,$$
$$\nabla_{X_2} = \partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_2\partial_\lambda + A_2$$

(here $A_1$, $A_2$ do not depend on $\lambda$). Lax pairs of this structure were already present in Zakharov and Shabat (1979).

The commutator of two covariant vector fields contains vector field part and Lie algebraic part,

$$[\nabla_{X_1}, \nabla_{X_2}] = [X_1, X_2] + X_1 A_2 - X_2 A_1 + [A_1, A_2]$$

Demanding both parts to be equal to zero, from the first part we get the system describing conformally ASD metric, and the second part gives the system for $A_1$, $A_2$

$$\partial_x A_2 = \partial_y A_1,$$
$$(\partial_z + F_x\partial_x + G_x\partial_y)A_2 - (\partial_w + F_y\partial_x + G_y\partial_y)A_1 + [A_1, A_2] = 0.$$

# ASDYM case

For $F = G = 0$ we have

$$X_1 = \partial_z - \lambda\partial_x,$$
$$X_2 = \partial_w - \lambda\partial_y,$$

$$\frac{1}{2}g = dwdx - dzdy.$$

The extended Lax pair takes the form

$$\nabla_{X_1} = \partial_z - \lambda\partial_x + A_1,$$
$$\nabla_{X_2} = \partial_w - \lambda\partial_y + A_2,$$

and the commutativity condition is

$$\partial_x A_2 = \partial_y A_1,$$
$$\partial_z A_2 - \partial_w A_1 + [A_1, A_2] = 0.$$

This a well known form of ASDYM equations for constant metric $g$ in a special gauge.

# General case

## 1. Geometry

*Extended Lax pair gives a general form of ASDYM equations for arbitrary conformally ASD metric g in signature (2,2) (locally, up to transformations of coordinates and a gauge).*

## 2. Integrability

*Extended Lax pair belongs to the hierarchy which unites ASDYM hierarchy and generic 4-dimensional dispersionless hierarchy. Lax-Sato equations and dressing scheme can be constructed for this hierarchy.*

# Geometry

Given: conformally ASD metric $g$ with signature (2,2) (ASD conformal structure) and ASD gauge field with a connection form $\mathbf{A}$. The corresponding gauge curvature form is $\mathbf{F} = \mathrm{d}\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$, it satisfies the ASDYM equation

$$\mathbf{F} = - * \mathbf{F}$$

First step:
following Dunajski, Ferapontov and Kruglikov, we find local coordinates $(z, w, x, y)$ such that ASD conformal structure is locally represented by a metric

$$\frac{1}{2}g = dwdx - dzdy - F_y dw^2 - (F_x - G_y)dwdz + G_x dz^2,$$

Second step:
notice that for this metric due to ASDYM equation we have

$$F_{34} = 0,$$

where we have used inverse matrix to metric $g$ defined by symmetric bivector

$$\frac{1}{2}Q = \partial_w\partial_x - \partial_z\partial_y + F_y\,\partial_x^2 + (G_y - F_x)\,\partial_x\partial_y - G_x\,\partial_y^2$$

$\det g = \det Q = 1$ (for this metric $F^{12} = F_{34}$). Then it is possible to choose a gauge such that

$$A_3 = A_4 = 0,$$

and we have only two nontrivial gauge field components $A_1$, $A_2$.

Third step:
we will prove that ASDYM equations for $A_1$, $A_2$ for the metric $g$ coincide with Lie algebraic part of compatibility equations for extended Lax pair.

# Tetrad of one-forms

The conformal structure is represented by (DFK)

$$g = 2(\mathbf{e}^{00'}\mathbf{e}^{11'} - \mathbf{e}^{01'}\mathbf{e}^{10'}),$$

where the tetrad of one-forms is

$$\begin{aligned}
\mathbf{e}^{00'} &= dw, \\
\mathbf{e}^{10'} &= dz, \\
\mathbf{e}^{01'} &= dy - G_y dw - G_x dz, \\
\mathbf{e}^{11'} &= dx - F_y dw - F_x dz.
\end{aligned}$$

# Tetrad of vector fields

The dual tetrad of vector fields is

$$
\begin{aligned}
\mathbf{e}_{00'} &= \partial_w + F_y\partial_x + G_y\partial_y, \quad (+A_2)\\
\mathbf{e}_{10'} &= \partial_z + F_x\partial_x + G_x\partial_y, \quad (+A_1)\\
\mathbf{e}_{01'} &= \partial_y,\\
\mathbf{e}_{11'} &= \partial_x,
\end{aligned}
$$

symmetric bivector reads

$$Q = 2(\mathbf{e}_{00'}\mathbf{e}_{11'} - \mathbf{e}_{01'}\mathbf{e}_{10'}).$$

ASDYM equations for this tetrad take the form

$$F_{00'10'} = 0, \quad F_{00'11'} = F_{10'01'}$$

For gauge field curvature **F** in the tetrad basis we use a standard formula

$$\mathbf{F}(\mathbf{u},\mathbf{v}) = \nabla_{\mathbf{u}}\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}} - \nabla_{[\mathbf{u},\mathbf{v}]}$$

for arbitrary vector fields **u**, **v**. Taking into account the structure of tetrade and the fact that for our gauge $A_3 = A_4 = 0$, we see that the third term doesn't contain a gauge field, and for the curvature components we get

$$F_{00'10'} = (\partial_w + F_y\partial_x + G_y\partial_y)A_1 - (\partial_z + F_x\partial_x + G_x\partial_y)A_2 - [A_1, A_2],$$
$$F_{00'11'} = -\partial_x A_2, \quad F_{10'01'} = -\partial_y A_1$$

Thus ASDYM equations read

$$(\partial_w + F_y\partial_x + G_y\partial_y)A_1 - (\partial_z + F_x\partial_x + G_x\partial_y)A_2 - [A_1, A_2] = 0,$$
$$\partial_x A_2 = \partial_y A_1,$$

which coincides with the Lie algebraic part of commutativity condition for extended vector fields Lax pair.

# Gauge-invariant SDYM equations

Lax pair

$$\nabla_{X_1} = \partial_z + F_x\partial_x + G_x\partial_y + A_1 - \lambda(\partial_x + B_1) + f_1\partial_\lambda,$$
$$\nabla_{X_2} = \partial_w + F_y\partial_x + G_y\partial_y + A_2 - \lambda(\partial_y + B_2) + f_2\partial_\lambda$$

Compatibility condition (matrix part)

$$\begin{aligned}
&\partial_x B_2 - \partial_y B_1 + [B_1, B_2] = 0,\\
&\partial_x A_2 - (\partial_w + F_y\partial_x + G_y\partial_y)B_1 - [B_1, A_2]\\
&\qquad = \partial_y A_1 - (\partial_z + F_x\partial_x + G_x\partial_y)B_2 - [B_2, A_1],\\
&(\partial_z + F_x\partial_x + G_x\partial_y)A_2 - (\partial_w + F_y\partial_x + G_y\partial_y)A_1\\
&\qquad + [A_1, A_2] + f_2 B_1 - f_1 B_2 = 0.
\end{aligned}$$

Represent ASDYM equations for the ASD conformal structure

$$\frac{1}{2}g = dwdx - dzdy - F_y dw^2 - (F_x - G_y)dwdz + G_x dz^2,$$

# Matrix dressing on the geometric background

Generally, we may consider matrix RH problem

$$\Phi_+ = \Phi_- R(\psi_1, \psi_2, \psi_3),$$

defined on some oriented curve $\gamma$ in the complex plane, or matrix $\bar\partial$ problem

$$\bar\partial \Phi = \Phi R(\psi_1, \psi_2, \psi_3),$$

defined in some region $G$, and $\psi_i(\lambda, \mathbf{t})$ are arbitrary wave functions of dispersionless Lax pair

$$X_1\psi_i = (\partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_1\partial_\lambda)\psi_i = 0,$$
$$X_2\psi_i = (\partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_2\partial_\lambda)\psi_i = 0,$$

defined on $\gamma$ or in $G$.

Let us suggest the existence of solution $\Phi$ of RH (or $\bar{\partial}$) problem having no zeroes and normalized by 1 at infinity. Then $X_1\Phi$, $X_2\Phi$ satisfy the same problem ($[X_1, R] = [X_2, R] = 0$), and the functions

$$(X_1\Phi)\Phi^{-1}, \ (X_2\Phi)\Phi^{-1}$$

are holomorphic in the complex plane.
Considering the behaviour at infinity, we get

$$(X_1\Phi)\Phi^{-1} = \partial_x \Phi_1(\mathbf{t}),$$
$$(X_2\Phi)\Phi^{-1} = \partial_y \Phi_1(\mathbf{t}),$$

or the solution for the extended Lax pair with the gauge field

$$A_1 = \partial_x \Phi_1(\mathbf{t}), \ A_2 = \partial_y \Phi_1(\mathbf{t}).$$

Dropping the normalization condition at infinity, we will get solution for gauge-invariant extended Lax pair.

There are important reductions connected with existence of *polynomial wave functions* for dispersionless Lax pair $\psi = P^n(\lambda)$, coefficients of the polynomial depends on times. A class of special ASDYM solutions for these geometries is defined by the problems

$$\Phi_+ = \Phi_- R(P^n) \quad \text{or}$$
$$\bar{\partial}\Phi = \Phi R(P^n).$$

Another important reduction of geometry: *linearly-degenerate case* (no $\partial_\lambda$ in dispersionless Lax pair, $\lambda$ is one of the wave functions),

$$\Phi_+ = \Phi_- R(\lambda, \psi_1, \psi_2) \quad \text{or}$$
$$\bar{\partial}\Phi = \Phi R(\lambda, \bar{\lambda}, \psi_1, \psi_2).$$

In this case ASDYM Lax pair admits rational (in $\lambda$) solutions with simple stationary poles (correspond to $\delta$-functions in the $\bar{\partial}$ problem), which can be calculated explicitly.

# From the dressing scheme to the hierarchy

1. **Dressing for vector fields.** Nonlinear vector Riemann-Hilbert problem (e.g. on the unit circle, here we don't discuss the question of reductions)

$$\begin{aligned} \Psi^0_{\text{in}} &= F_0(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \Psi^2_{\text{out}}), \\ \Psi^1_{\text{in}} &= F_1(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \Psi^2_{\text{out}}), \\ \Psi^2_{\text{in}} &= F_2(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \Psi^2_{\text{out}}), \end{aligned}$$

the expansions at infinity are

$$\Psi^0_{\text{out}} = \lambda + \sum_{n=1}^{\infty} \Psi^0_n(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n},$$

$$\Psi^1_{\text{out}} = \sum_{n=0}^{\infty} t^1_n (\Psi^0)^n + \sum_{n=1}^{\infty} \Psi^1_n(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n}$$

$$\Psi^2_{\text{out}} = \sum_{n=0}^{\infty} t^2_n (\Psi^0)^n + \sum_{n=1}^{\infty} \Psi^2_n(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n},$$

inside the unit circle the functions are analytic.

$\Psi^0$, $\Psi^1$, $\Psi^2$ will give the wave fuctions for the hierarchy of commuting vector fields, defined through coefficients of expansion of these functions.

2. **Matrix dressing on the background.** Consider a matrix Riemann-Hilbert problem

$$\Phi_{\text{in}} = \Phi_{\text{out}} R(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \Psi^2_{\text{out}}),$$

$\Phi$ is normalized by 1 at infinity and analytic inside and outside the unit circle,

$$\Phi_{\text{out}} = 1 + \sum_{n=1}^{\infty} \Phi_n(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n}$$

Expansions of $\Psi$, $\Phi$ give coefficients for extended Lax pair, $\Phi$ is a wave function. A general wave function is given by the expression $\Phi F(\Psi^0, \Psi^1, \Psi^2)$, $F$ is arbitrary matrix function.
For constant metric $g$ corresponding to trivial vector fields we have

$$\Psi^0 = \lambda, \quad \Psi^1 = x + \lambda z, \quad \Psi^2 = y + \lambda w,$$

and we get standard Riemann-Hilbert problem for ASDYM.

The vector fields part of the dressing scheme implies analyticity in the complex plane of the form (no dicontinuity on the unit circle)

$$\omega = \left| \frac{D(\Psi^0, \Psi^1, \Psi^2)}{D(\lambda, x_1, x_2)} \right|^{-1} \mathrm{d}\Psi^0 \wedge \mathrm{d}\Psi^1 \wedge \mathrm{d}\Psi^2,$$

where $x_1 = t_0^1$, $x_2 = t_0^2$ are lowest times of the hierarchy, and from matrix Riemann problem we get analyticity of the matrix-valued form

$$\Omega = \omega \wedge \mathrm{d}\Phi \cdot \Phi^{-1}.$$

Analyticity of these forms imply the relations

$$(\omega_{\text{out}})_- = \left( \left| \frac{D(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \Psi^2_{\text{out}})}{D(\lambda, x_1, x_2)} \right|^{-1} \mathrm{d}\Psi^0_{\text{out}} \wedge \mathrm{d}\Psi^1_{\text{out}} \wedge \mathrm{d}\Psi^2_{\text{out}} \right)_- = 0,$$

$$(\Omega_{\text{out}})_- = (\omega_{\text{out}} \wedge \mathrm{d}\Phi_{\text{out}} \cdot \Phi^{-1}_{\text{out}})_- = 0$$

for the series $\Psi^0_{\text{out}}$, $\Psi^1_{\text{out}}$, $\Psi^2_{\text{out}}$, $\Phi_{\text{out}}$. These relations are generating relations for the hierarchy in terms of formal series, they are equivalent to the complete set of Lax-Sato equations of the hierarchy.

First relation gives Lax-Sato equations for the hierarchy of commuting polynomial in $\lambda$ vector fields (here we drop subscript 'out' for the series):

$$\partial^k_n \Psi = \sum_{i=0}^{2} \left( \left( \frac{D(\Psi^0, \Psi^1, \Psi^2)}{D(\lambda, x_1, x_2)} \right)^{-1}_{ik} (\Psi^0)^n \right)_+ \partial_i \Psi,$$

where $1 \leqslant n < \infty$, $k = 1, 2$, $\partial_0 = \partial_\lambda$, $\partial_1 = \partial_{x_1}$, $\partial_2 = \partial_{x_2}$,
$\Psi = (\Psi^0, \Psi^1, \Psi^2)$.

The second generating relation gives Lax-Sato equations for $\Phi$ on the vector field background in terms of extended polynomial vector fields,

$$\partial_n^k \Psi = V_n^k(\lambda)\Psi,$$
$$\partial_n^k \Phi = \left( V_n^k(\lambda) - ((V_n^k(\lambda)\Phi) \cdot \Phi^{-1})_+ \right) \Phi$$

First flows give exactly the extended Lax pair for ASDYM equations on ASD background, if we identify $z = t_1^1$, $w = t_1^2$, $x = x_1$, $y = x_2$.

# Questions

- Solutions!
- Higher-dimensional case – what is the geometry?
- Lower-dimensional cases and reductions – known integrable systems on the background?

THANK YOU!